DISHA CLASSES
Guiding you to Success

Class test - 18

(61) $\int_a^b \frac{x}{|x|}dx$

Case (I) $\quad 0 < a < b \Rightarrow \int_a^b 1\cdot dx = b - a \quad — (1)$

Case II $\quad a < b < 0 \Rightarrow \int_a^b -1 dx = -(b-a) \quad — (2) = |b| - |a|$

Case III $\quad a < 0 < b \Rightarrow \int_a^b \frac{x}{|x|}dx = \int_a^0 -1 dx + \int_0^b 1\cdot dx$

$= a + b \quad — (3) = |b| - (-a) = |b| - |a|$

From all cases $\Rightarrow \int_a^b \frac{x}{|x|}dx = |b| - |a|$

(62) $\int_0^{4/\pi} \underset{2^{nd}}{3x^2} \underset{1^{st}}{\sin 1/x}\, dx - \int_0^{4/\pi} x \cos 1/x \, dx$

$= x^3 \sin 1/x \Big|_0^{4/\pi} - \int_0^{4/\pi} x^3 \cos 1/x \left(-1/x^2\right) dx - \int_0^{4/\pi} x \cos 1/x \, dx$

$= \frac{64}{\pi^3} \sin \pi/4 + \int_0^{4/\pi} x \cos 1/x\, dx - \int_0^{4/\pi} x \cos 1/x \, dx$

$= \frac{64}{\pi^3} \cdot \frac{1}{\sqrt{2}}$

(63) $I = \int_0^{\pi/2n} \frac{dx}{1 + \cot^n nx} \quad — (1)$

$I = \int_0^{\pi/2n} \frac{dx}{1 + \cot^n n(\pi/2n - x)} = \int_0^{\pi/2n} \frac{dx}{1 + \cot^n(\pi/2 - nx)}$

$= \int_0^{\pi/2n} \frac{dx}{1 + \tan^n nx} = \int_0^{\pi/2n} \frac{dx}{1 + \frac{1}{\cot^n nx}} = \int_0^{\pi/2n} \frac{\cot^n nx}{\cot^n nx + 1} d \quad — (2)$

$(1) + (2) \Rightarrow 2I = \int_0^{\pi/2n} 1\cdot dx = \pi/2n$

$I = \pi/4n$

(64) $\int_0^{\pi/2} \frac{4\cos^3 x - 3\cos x + 1}{2\cos x - 1} dx$

$= \int_0^{\pi/2} (2\cos^2 x + \cos x - 1)\, dx$

$= \int_0^{\pi/2} (1 + \cos 2x) dx + \int_0^{\pi/2} \cos x\, dx - \int_0^{\pi/2} 1\cdot dx$

$= \pi/2 + 0 + (1) - \pi/2 = +1$

$$
\begin{array}{r|l}
 & 2\cos^2 x + \cos x - 1 \\
\hline
2\cos x - 1 & 4\cos^3 x - 3\cos x + 1 \\
 & 4\cos^3 x - 2\cos^2 x \\
 & \;\;-\quad\;\; + \\
\hline
 & +2\cos^2 x - 3\cos x \\
 & 2\cos^2 x - \cos x \\
 & \;-\quad\;\; + \\
\hline
 & -2\cos x + 1 \\
 & -2\cos x + 1 \\
 & +\quad\;\; - \\
\hline
 & 0
\end{array}
$$

(65) $C_1 \to C_1 - C_2 - C_3$

$$f(x) = \begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ 0 & 3 & 4\sin x \\ 0 & \sin x & 1 \end{vmatrix} = \sin x(3 - 4\sin^2 x) = \sin 3x$$

$$\int_0^{\pi/2} \sin 3x\, dx = -\frac{1}{3}\cos 3x \Big|_0^{\pi/2} = -\frac{1}{3}[0-1] = \frac{1}{3}$$

(66) $\int_{-1}^{2}(2-x)\,dx + \int_{2}^{3}(x-2)\,dx + \int_{-1}^{0} -1\,dx + \int_0^1 0\,dx + \int_1^2 1\cdot dx + \int_2^3 2\cdot dx$

$$= 2x - \frac{x^2}{2}\Big|_{-1}^{2} + \frac{x^2}{2} - 2x\Big|_2^3 \cancel{-}1 + 0 + 1 + 2$$

$$= 2 - \left(-\frac{5}{2}\right) + \left(-\frac{3}{2} - (-2)\right) + \cancel{2} = \cancel{2} + \frac{9}{2} + \frac{1}{2} = \cancel{8}\ 7$$

(67) $$\int \frac{(x^{10})^{1/2}\sqrt{5/x^{10}+1}}{x^{16}}\,dx = \int \frac{(5\cdot x^{-10}+1)^{1/2}}{x^{11}}\,dx$$

Sub $5x^{-10} + 1 = t \Rightarrow -\frac{50}{x^{11}}dx = dt$

$$= \int t^{1/2}\left(-\frac{1}{50}dt\right) = -\frac{1}{\frac{50\times 3}{2}}\, t^{3/2} + c$$

(68) $$\int_0^1 \frac{\ln(1+x^2)}{1+x^2}\,dx \qquad \text{sub } x = \tan\theta$$

$$\int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)$$

Now replace x by $\pi/4 - \theta$

(69) $\int \frac{x^8}{x^{12}-1}dx = \int \frac{(x^3)^2 \; x^2}{(x^3)^4-1}dx$

let $x^3 = t \Rightarrow 3x^2\,dx = dt$

$= \frac{1}{3}\int \frac{t^2\,dt}{t^4-1} = \frac{1}{3}\int \frac{t^2}{(t^2-1)(t^2+1)}dt$

$= \frac{1}{6}\int \frac{2t^2}{(t^2-1)(t^2+1)}dt$

$= \frac{1}{6}\int \frac{\overline{t^2+1} + \overline{t^2-1}}{(t^2-1)(t^2+1)}dt$

$= \frac{1}{6}\int\left(\frac{1}{t^2-1} + \frac{1}{t^2+1}\right)dt$

$= \frac{1}{6}\times\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + \frac{1}{6}\tan^{-1}t$

$= \frac{1}{12}\ln\left|\frac{x^3-1}{x^3+1}\right| + \frac{1}{6}\tan^{-1}x^3 + C$

(70) $\frac{1}{(x^2+a^2)(x^2+b^2)}$ let $x^2 = t$

$\Rightarrow \frac{1}{(t+a^2)(t+b^2)} = \frac{A}{t+a^2} + \frac{B}{t+b^2}$

$A = \frac{1}{b^2-a^2}$ & $B = \frac{1}{a^2-b^2}$

$\Rightarrow \frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(b^2-a^2)}\cdot\frac{1}{x^2+a^2} + \frac{1}{a^2-b^2}\cdot\frac{1}{x^2+b^2}$

$\Rightarrow \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\cdot\frac{1}{a}\tan^{-1}\frac{x}{a}\Big|_0^\infty + \frac{1}{(a^2-b^2)\cdot b}\tan^{-1}\frac{x}{b}\Big|_0^\infty$

$= \frac{1}{a(b^2-a^2)}\cdot\frac{\pi}{2} + \frac{\pi/2}{b(a^2-b^2)}$

$= \frac{\pi/2}{a^2-b^2}\left[\frac{1}{b}-\frac{1}{a}\right] = \frac{\pi}{2(a+b)ab}$

(71) $\displaystyle\int_{-\pi/3}^{\pi/3} x\tan x\sec^2 x\,dx = 2\int_0^{\pi/3} x\tan x\sec x\,dx$ (even functi

First function, 2nd function

Integrate by parts

(72) $\displaystyle\int_{1/e}^{\tan x}\frac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x}\frac{dt}{t(1+t^2)}$ Sub $t = \frac{1}{u}$

$$+\int_e^{\tan x}\frac{-\frac{1}{u^2}\,du}{\frac{1}{u}\left(1+\frac{1}{u^2}\right)}$$

$$+\int_e^{\tan x}\frac{-u\,du}{u^2+1}$$

$$\int_{1/e}^{\tan x}\frac{t\,dt}{1+t^2} + \int_{\tan x}^{e}\frac{u\,du}{u^2+1}$$

$$\int_{1/e}^{\tan x}\frac{t\,dt}{1+t^2} + \int_{\tan x}^{e}\frac{t\,dt}{t^2+1}$$ (replacing u by t)

$$= \int_{1/e}^{e}\frac{\cancel{2}t\,dt}{t^2+1} = \frac{1}{2}\ln(t^2+1)\Big|_{1/e}^{e}$$

$$= \frac{1}{2}\left[\ln(e^2+1) - \ln\left(\frac{1}{e^2}+1\right)\right]$$

$$= \frac{1}{2}\left[\ln(e^2+1) - \ln(1+e^2) + \ln e^2\right]$$

$$= 1$$

(73) $\cancel{x}$ $\displaystyle\int\frac{dx}{(x^2)^{3/2}\left(\frac{1}{x^2}+1\right)^{3/2}} = \int\frac{dx}{x^3\left(\frac{1}{x^2}+1\right)^{3/2}}$

Sub. $\frac{1}{x^2}+1 = t$

$-\frac{2}{x^3}dx = dt$

$$= \int\frac{-\frac{1}{2}dt}{t^{3/2}} = -\frac{1}{2}\,\frac{t^{-1/2}}{-\frac{1}{2}} + c$$

$$= \frac{1}{t^{1/2}} + c$$

$$= \frac{1}{\left(\frac{1}{x^2}+1\right)^{1/2}} + c = \frac{x}{(1+x^2)^{1/2}} + c$$

$y(0) = 0 \Rightarrow c = 0.$

$\Rightarrow$ so $y = \dfrac{x}{(1+x^2)^{1/2}} \Rightarrow y(1) = \dfrac{1}{\sqrt{2}}$

(74) $\displaystyle \lim_{x\to\infty} \frac{\left(\int_0^x e^x dx\right)^2}{\int_0^x e^{2x^2} dx} = \frac{2\left(\int_0^x e^x dx\right)\cdot e^x}{e^{2x^2}\,\cancel{dx}}$ (Applying L.H. rule)

$\displaystyle = \lim_{x\to\infty} \frac{2\int_0^x e^x dx}{e^{2x^2-x}} = \lim_{x\to\infty} \frac{2e^x}{e^{2x^2-x}(4x-1)}$

$\displaystyle = \lim_{x\to\infty} \frac{2}{e^{2x^2-2x}(4x-1)} = \frac{2}{\infty}$

$= 0$

(75) $\displaystyle \int \frac{dx}{x\sqrt{1-x^3}} = \int \frac{x^2\,dx}{x^3\sqrt{1-x^3}}$

let $1-x^3 = t^2 \Rightarrow -3x^2 dx = 2t\,dt$

$\displaystyle = -\frac{2}{3}\int \frac{t\,dt}{(1-t^2)\,t} = \frac{2}{3}\int \frac{dt}{t^2-1}$

$\displaystyle = \frac{2}{3} \times \frac{1}{2} \ln\left|\frac{t-1}{t+1}\right| + c$

(76) $\displaystyle \int \frac{1-x^7}{x(1+x^7)}\,dx = \int \frac{2-(x^7+1)}{x(x^7+1)}\,dx$

$\displaystyle = 2\int \frac{dx}{x(x^7+1)} - \int \frac{1}{x}\,dx$

$\displaystyle = 2\int \frac{dx}{x^8(1+x^{-7})} - \ln x + c$

sub $1 + x^{-7} = u$

$-7/x^8\,dx = du$

$\displaystyle = -\frac{2}{7}\int \frac{du}{u} - \ln x + c$

$\displaystyle = -\frac{2}{7}\ln\left(1+\frac{1}{x^7}\right) - \ln x + c$

$\displaystyle = -\frac{2}{7}\ln(x^7+1) + \frac{2}{7}\ln x^7 - \ln x + c$

$\displaystyle = -\frac{2}{7}\ln(x^7+1) + 2\ln x - \ln x + c$

(77) $\int_0^{\pi} \frac{dx}{1-2a\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)+a^2} = \int_0^{\pi} \frac{\sec^2 x/2\, dx}{1+\tan^2 x/2 - 2a(1-\tan^2 x/2) + a^2}$

$= \int_0^{\pi} \frac{\sec^2 x/2\, dx}{(1-2a+a^2)+(1+2a+a^2)\tan^2 x/2} = \int_0^{\pi} \frac{\sec^2 x/2\, dx}{(1-a)^2+(1+a)^2\tan^2 x/2}$

$= \frac{1}{(1+a)^2}\int_0^{\pi} \frac{\sec^2 x/2\, dx}{\left(\frac{1-a}{1+a}\right)^2+\tan^2 x/2}$

let $\tan x/2 = t \Rightarrow \frac{1}{2}\sec^2 x/2\, dx = dt$

$= \frac{1}{(1+a)^2}\int_0^{\infty} \frac{2\, dt}{\left(\frac{1-a}{1+a}\right)^2+t^2}$

$= \frac{2}{(1+a)^2\cdot\frac{(1-a)}{(1+a)}} \tan^{-1}\left(\frac{t}{\frac{1-a}{1+a}}\right)\Bigg|_0^{\infty}$

$= \frac{2}{(1-a)(1+a)}\left[\tan^{-1}\left(\frac{1+a}{1-a}\right)t\right]_0^{\infty}$

if $-1<a<1$ then $\frac{1+a}{1-a}$ is +ive

$= \frac{2}{1-a^2}\left[\tan^{-1}\infty - \tan^{-1}0\right]$

$= \frac{2}{(1-a^2)}\cdot\pi/2 = \frac{\pi}{1-a^2}$

(78) $I_1 = \tan^{-1}t\Big|_x^1 = \pi/4 - \tan^{-1}x$

$I_2 = \tan^{-1}t\Big|_1^{1/x} = \cancel{\pi/4 - \tan^{-1}\frac{1}{x} - \pi/4}\ \cot^{-1}$

$= \tan^{-1}1/x - \pi/4 = \cot^{-1}x - \pi/4$

$= \pi/2 - \tan^{-1}x - \pi/4$

$\Rightarrow$ a, d both are correct.

(79) $g'(x) = x f'(x)$, since $f(x)$ is increasing

$\Rightarrow f'(x) > 0$

so $g'(x) > 0$, $x>0$ & $g'(x) < 0$ when $x<0$

$\Rightarrow$ (c) & (d) are correct.

(80) $\int_1^5 [|x-3|]dx = \int_1^2 1\cdot dx + \int_2^3 0\, dx + \int_3^4 0\, dx + \int_4^5 1\, dx$

$= 1 + 0 + 0 + 1 = 2$

(81) $f(-x) = \begin{vmatrix} -x & \cos(-x) & e^{x^2} \\ \sin(-x) & (-x)^2 & \sec(-x) \\ \tan(-x) & 1 & 2 \end{vmatrix}$

$= -\begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix} = -f(x)$

$\Rightarrow f(x)$ is odd function

so $\int_{-\pi/2}^{\pi/2} f(x)dx = 0$

(82) $I = \int_4^{10} \frac{[x^2]}{[(x-14)^2] + [x^2]} dx \quad -①$

$I = \int_4^{10} \frac{[(14-x)^2]}{[(14-x-14)^2] + [(14-x)^2]}$

$I = \int_4^{10} \frac{[(x-14)^2]}{[x^2] + [(x-14)^2]} \quad -②$

Adding ① & ②

$\Rightarrow 2I = \int_4^{10} \frac{[x^2] + [(x-14)^2]}{[x^2] + [(x-14)^2]} dx$

$= \int_4^{10} dx = x\Big|_4^{10} = 6$

$I = 3$

(83) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r+n}{n^2+r^2} = \lim_{n\to\infty} \frac{n}{n^2}\left(\frac{r/n + 1}{1+\left(\frac{r}{n}\right)^2}\right)$

$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{r/n+1}{1+r^2/n^2}\right)$

$= \int_0^1 \frac{x+1}{1+x^2} dx$

(84) $a f(x) + b f(1/x) = \frac{1}{x} - 5$ — (1)

replace $x \to 1/x$

$\Rightarrow a f(1/x) + b f(x) = x - 5$ — (2)

(1) $\times a$ − (2) $\times b$

$\Rightarrow (a^2 - b^2) f(x) = \frac{a}{x} - 5a - bx + 5b$

$\Rightarrow f(x) = \frac{1}{a^2-b^2}\left[\frac{a}{x} - bx + 5(b-a)\right]$

$\int_1^2 f(x)\,dx = \frac{1}{a^2-b^2}\left[a \ln 2 - \frac{3b}{2} + 5(b-a)\right]$

(85) $\int_0^1 t f''(t)\,dt = 0$

$\Rightarrow t f'(t)\Big|_0^1 - \int_0^1 1 \times f'(t)\,dt = 0$

$= f'(1) - 0 - \int_0^1 f'(t)\,dt = 0$

$\Rightarrow f'(1) - [f(t)]\Big|_0^1 = 0$

$\Rightarrow f'(1) - [f(1) - f(0)] = 0$

$\Rightarrow 1 - [f(1) - 1] = 0 \Rightarrow f(1) = 2$

(86) $F(t) = \int_0^t e^{t-y}\, y\, dy$

$= -y e^{t-y}\Big|_0^t - \int_0^t 1\,(-e^{t-y})\,dy$

$= -t + (-e^{t-y})\Big|_0^t$

$= -t + [-e^0 - (-e^{t})]$

$= -t + [-1 + e^t] = e^t - t - 1$

(87) $\int_0^{\pi/4} (1 - 2\sin^2 x)^{3/2}\, \cancel{\sin x} \cos x\, dx$

$\int_0^{\pi/4} \left[1 - (\sqrt{2}\sin x)^2\right]^{3/2} \cos x\, dx$

$\sqrt{2}\sin x = \sin\theta$

$\sqrt{2}\cos x\, dx = \cos\theta\, d\theta$

$\int_0^{\pi/2} (1 - \sin^2\theta)^{3/2} \frac{\cos\theta\, d\theta}{\sqrt{2}}$

$$\frac{1}{\sqrt2}\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{1}{\sqrt2}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{4\sqrt2}\int_0^{\pi/2}\left(1+2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{1}{4\sqrt2}\int_0^{\pi/4}\left[1+2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{4\sqrt2}\left[\int_0^{\pi/2} \frac{3}{2}\, d\theta + \int_0^{\pi/2} 2\cos 2\theta \, d\theta + \frac{1}{2}\int_0^{\pi/2}\cos 4\theta \, d\theta\right]$$

$$= \frac{1}{4\sqrt2}\left[\frac{3}{2}\frac{\pi}{2} + 0 + 0\right]$$

$$= \frac{3\pi}{8\sqrt2 \times 2} = \frac{3\pi}{16\sqrt2}$$

(88)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^{1/a}\, n^{a-1/a}\left[1+\left(\frac{k}{n}\right)^{a-1/a}\right]}{n^a \cdot n}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^{1/a}\left[1+\left(\frac{k}{n}\right)^{a-1/a}\right]$$

$$= \int_0^1 x^{1/a}\left(1+x^{a-1/a}\right) dx$$

$$= \int_0^1 \left(x^{1/a} + x^a\right) dx = \frac{1}{1+1/a} + \frac{1}{1+a}$$

$$= 1$$

(89)

$$I = \int_0^1 \frac{dx}{(1-x+x^2)\left(e^{2x-1}+1\right)} \quad \text{— (1)}$$

$$I = \int_0^1 \frac{dx}{\left[1-(1-x)+(1-x)^2\right]\left(e^{2(1-x)-1}+1\right)}$$

$$= \int_0^1 \frac{dx}{(1-x+x^2)\left(e^{-(2x-1)}+1\right)}$$

$$= \int_0^1 \frac{dx}{(1-x+x^2)\left[\frac{1}{e^{2x-1}}+1\right]} = \int_0^1 \frac{e^{2x-1}\,dx}{(1-x+x^2)\left(1+e^{2x-1}\right)} \quad \text{— (2)}$$

(1) + (2)

$$\Rightarrow 2I = \int_0^1 \frac{(e^{2x-1}+1)}{(1-x+x^2)(e^{2x-1}+1)}dx$$

$$I = \frac{1}{2}\int_0^1 \frac{dx}{1-x+x^2}$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{2}\times\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right)$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\pi/6 + \pi/6\right] = \frac{\pi}{3\sqrt{3}}$$

(30) $A = \int_0^{2\pi}\sqrt{1-\sin x}\,dx = \int_0^{2\pi}\sqrt{1-\sin(2\pi-x)}\,dx$

$$= \int_0^{2\pi}\sqrt{1+\sin x}\,dx = B$$

$C = \int_0^{2\pi}\sqrt{1-\sin 2x}\,dx = \int_0^{2\pi}\sqrt{1+\sin 2x}\,dx$ (~~Applying~~

$= D$ (replacing x by $2\pi - x$)

In $C = \int_0^{2\pi}\sqrt{1-\sin 2x}\,dx$

let $2x = t$

$$= \int_0^{4\pi}\sqrt{1-\sin t}\,\frac{dt}{2} = \frac{1}{2}\int_0^{4\pi}\sqrt{1-\sin t}\,dt$$

$$= \frac{1}{2}\times 2\int_0^{2\pi}\sqrt{1-\sin t}\,dt$$

(Periodic of period 2π)

$$= \int_0^{2\pi}\sqrt{1-\sin t}\,dt$$

$= A = B$

$\Rightarrow A = B = C = D$