DISHA CLASSES
Guiding you to Success

Phase Test - 5 (Solutions)

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(61) (c) $f'(x) = 2\cos x + 2\cos 2x = 0$

$\Rightarrow \cos 2x + \cos x = 0 \Rightarrow 2\cos^2 x - 1 + \cos x = 0$

$\Rightarrow 2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow 2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{1}{2}, -1$

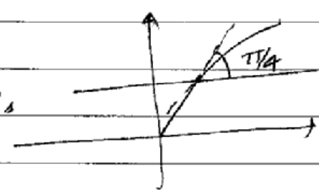
$\Rightarrow x = \pi/3, \pi$ in $x \in [0, 3\pi/2]$

$f(0) = 0$, $f(3\pi/2) = 2$, $f(\pi/3) = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$f(\pi) = 0$

$\Rightarrow f_{max} = f(\pi/3) = \frac{3\sqrt{3}}{2}$

(62) (d) At point of intersection tangent makes 45° with x axis



$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \tan 45^\circ$

$\Rightarrow x = \frac{1}{4}$

so Pt. of intersection $\left(\frac{1}{4}, \frac{1}{2}\right)$

Eqn of line $y = \frac{1}{2}$

(63) $3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{6x}{12-3y^2} = \frac{2x}{4-y^2}$

For vertical tangent $\frac{dy}{dx} = \infty \Rightarrow 4 - y^2 = 0$
$y = \pm 2$

If $y = 2 \Rightarrow 2^3 + 3x^2 = 12 \times 2$
$x = \pm \frac{4}{\sqrt{3}}$

If $y = -2 \Rightarrow -8 + 3x^2 = 12 \times (-2)$
$\Rightarrow$ no real x so

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~~only points are~~ $\left(\pm \frac{4}{\sqrt{5}}, 2\right)$

(64) (d) $f(x)\, f'(x) < 0 \Rightarrow \frac{d}{dx}\left\{\frac{1}{2}[f(x)]^2\right\} < 0$

$\Rightarrow [f(x)]^2$ is decreasing curve

if $[f(x)]^2$ is decreasing $\Rightarrow |f(x)|$ is also decreasing.

(65) (b) $xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} < 0, \ \forall x \in R$

$\Rightarrow$ Slope of normal $= -\frac{1}{\frac{dy}{dx}} = x^2 > 0$

so at any point on the curve normal has positive slope

slope of normal (from eqn) $= -\frac{a}{b} > 0$

$\Rightarrow$ a & b are opposite in sing

$\Rightarrow ab < 0$

(66) (a) $f'(x) = \frac{\cos x}{x} = 0 \Rightarrow x = (2n+1)\pi/2$

when $x = \pi/2,\ 5\pi/2,\ 9\pi/2 \ldots\ (n = 0, 2, 4, 6 \ldots)$

sign of $\cos x$ changes from +ive to −ive $\Rightarrow$ points of maxima

~~so maxima~~

when $x = 3\pi/2,\ 7\pi/2,\ 11\pi/2\ (n = 1, 3, 5 \ldots)$

$\cos x$ changes sign from −ive to +ive

$\Rightarrow$ points of minima.

when $x = -\pi/2, -5\pi/2, -9\pi/2 \ldots (n = -1, -3, -5 \ldots)$

$\frac{\cos x}{x}$: $+\ -$ about $x = -\pi/2$ $\Rightarrow$ points of maxima

$x = -3\pi/2, -7\pi/2, -11\pi/2\ (n = -2, -4, -6 \ldots)$ are points of minima.

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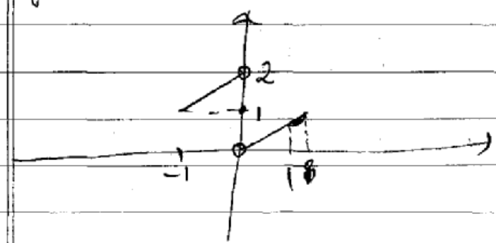
(67) (d) $f(x) = \begin{cases} x+2, & -1 \le x < 0 \\ 1, & x = 0 \\ x/2, & 0 < x \le 1 \end{cases}$

Graph of $f(x)$



$f(x)$ is discont. at $x=0$ but $x=0$ is neither maxima nor minima.

(68) (a) For first curve.
diff w.r.t. to x.
$3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2-y^2}{2xy} = m_1$

For second curve.
$6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2xy}{y^2-x^2} = m_2$

$m_1 m_2 = -1 \Rightarrow$ curves intersects at right angle.

(69) (a) $\frac{dy}{dx} = 3(a+2)x^2 - 6ax + 9a < 0, \forall x \in R$

$\Rightarrow a+2 < 0 \Rightarrow a < -2$

& $b^2 - 4ac < 0$ — (1)

$\Rightarrow 36a^2 - 4 \times 3(a+2) \times 9a < 0$

$\Rightarrow 36[a^2 - 3a(a+2)] < 0$

$\Rightarrow a[a - 3a - 6] < 0 \Rightarrow -a(2a+6) < 0$

$\Rightarrow a(a+3) > 0$

$\Rightarrow$ + (−3) − (0) +

$\Rightarrow a < -3$ or $a > 0$

Since from (1) $a < -2 \Rightarrow a < -3$

70 (c) $f'(x) = a^2 - 2a - 2 - \sin x < 0 \quad \forall\, x \in R.$

$f'_{max} = a^2 - 2a - 2 + 1$ should be less than zero if above conditions is satisfied.

$\Rightarrow a^2 - 2a - 1 < 0$

$[a - (1+\sqrt{2})][a - (1-\sqrt{2})] < 0$

$\Rightarrow 1 - \sqrt{2} < a < 1 + \sqrt{2}$

71 (a) $x^4 + y^4 = a^4$ let point be $(a\sqrt{\cos\theta}, a\sqrt{\sin\theta})$

compare with circle $x^2 + y^2 = a^2$

$4x^3 + 4y^3 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3} = \cancel{\sin^{3/2}\theta}$

$= -\frac{\cos^{3/2}\theta}{\sin^{3/2}\theta} = -\cot^{3/2}\theta$

Eqn. of tangent

$y - a\sqrt{\sin\theta} = -\cot^{3/2}\theta\,(x - a\sqrt{\cos\theta})$

For ~~point~~ x intercept $y = 0$

$\Rightarrow -a\sqrt{\sin\theta} = -\cot^{3/2}\theta\,(x - a\sqrt{\cos\theta})$

$\Rightarrow x = \frac{a\sqrt{\sin\theta}}{\cot^{3/2}\theta} + a\sqrt{\cos\theta}$

$= \frac{a\sin^2\theta + a\cos^2\theta}{\cancel{\cot}\cos^{3/2}\theta} = \frac{a}{\cos^{3/2}\theta}$

Similarly y ~~intercept~~ intercept $= \frac{a}{\sin^{3/2}\theta}$

so $p = \frac{a}{\cos^{3/2}\theta}, \quad q = \frac{a}{\sin^{3/2}\theta}$

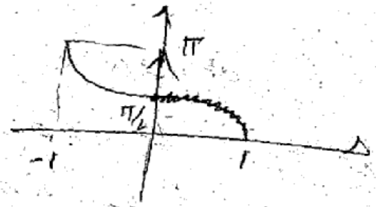
$p^{-4/3} + q^{-4/3} = a^{-4/3}\cos^2\theta + a^{-4/3}\sin^2\theta = a^{-4/3}$

72 (d) $\frac{x^2}{1+x^2} = y \Rightarrow x^2 = y + yx^2 \Rightarrow x^2 = \frac{y}{1-y} \geq 0$

$\Rightarrow \frac{y}{y-1} \leq 0$

$\Rightarrow 0 \leq y < 1$

so $0 \le \frac{x^2}{1+x^2} < 1$



when $0 \le x < 1$

$0 \le \cos^{-1}x \le \pi/2$

so $0 \le \frac{x^2}{1+x^2} < 1 \Rightarrow 0 < \cos^{-1}x \le \pi/2$

(73) (c) $a^2 + c^2 = b^2$, $a + b = 4$

Area $= \frac{1}{2}ac$ $= \frac{1}{2}a\sqrt{b^2 - a^2}$

$= \frac{1}{2}a\sqrt{(4-a)^2 - a^2}$

$= \frac{1}{2}a\sqrt{16 - 8a}$

$A^2 = \frac{1}{4}a^2[16 - 8a]$

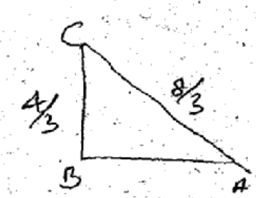
$A^2 = 2[2a^2 - a^3]$

$\frac{dA^2}{da} = 2[4a - 3a^2] = 0 \Rightarrow a = 4/3$

$\frac{d^2A^2}{da^2} = 2[4 - 6a] = 2[4 - 6 \times \frac{4}{3}] < 0$

$\Rightarrow$ area is maximum

so $a = \frac{4}{3}$, $b = 4 - 4/3 = 8/3$



$\sin A = \frac{4/3}{8/3} = \frac{1}{2}$

$\Rightarrow A = 30^\circ$

$\Rightarrow \angle C = 60^\circ$

(74) (a) $-x + ay + az = 0$

$bx - y + bz = 0$ have non trivial solution

$cx + cy - z = 0$

$\Rightarrow \begin{vmatrix} -1 & a & a \\ b & -1 & b \\ c & c & -1 \end{vmatrix} = 0$

$$\therefore \begin{vmatrix} \frac{-1}{1+a} & \frac{a}{1+a} & \frac{a}{1+a} \\ \frac{b}{1+b} & -\frac{1}{1+b} & \frac{b}{1+b} \\ \frac{c}{1+c} & \frac{c}{1+c} & -\frac{1}{1+c} \end{vmatrix} = 0$$

Since $\frac{a}{1+a} = \frac{a+1-1}{1+a} = 1 - \frac{1}{1+a}$

$$\Rightarrow \begin{vmatrix} \frac{-1}{1+a} & 1-\frac{1}{1+a} & 1-\frac{1}{1+a} \\ 1-\frac{1}{1+b} & -\frac{1}{1+b} & 1-\frac{1}{1+b} \\ 1-\frac{1}{1+c} & 1-\frac{1}{1+c} & -\frac{1}{1+c} \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 2-\frac{1}{1+a}-\frac{1}{1+b}-\frac{1}{1+c} & 2-\frac{1}{1+a}-\frac{1}{1+b}-\frac{1}{1+c} & 2-\frac{1}{1+a}-\frac{1}{1+b}-\frac{1}{1+c} \\ 1-\frac{1}{1+b} & -\frac{1}{1+b} & 1-\frac{1}{1+b} \\ 1-\frac{1}{1+c} & 1-\frac{1}{1+c} & -\frac{1}{1+c} \end{vmatrix} = 0$$

$$\Rightarrow \left(2-\frac{1}{1+a}-\frac{1}{1+b}-\frac{1}{1+c}\right) \begin{vmatrix} 1 & 1 & 1 \\ 1-\frac{1}{1+b} & -\frac{1}{1+b} & 1-\frac{1}{1+b} \\ 1-\frac{1}{1+c} & 1-\frac{1}{1+c} & -\frac{1}{1+c} \end{vmatrix} = 0$$

$$\Rightarrow 2-\frac{1}{1+a}-\frac{1}{1+b}-\frac{1}{1+c} = 0$$

75 (b) $A(\text{adj} A) = |A| I = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow |A| = 2$

$|\text{adj} A| = |A|^{3-1} = |A|^2 = 2^2$

76 (b) $AX = \lambda X \Rightarrow (A - \lambda I) X = 0$

$\Rightarrow$ homogenious system of eqn. in terms of x

Since $X \neq 0 \Rightarrow$ non trivial sol.

so $|A - \lambda I| = 0$.

matrix $A - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3 \Rightarrow \begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$\Rightarrow (2-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -(\lambda+1) & 0 \\ 1 & 0 & -(\lambda+1) \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(\lambda+1)^2 = 0 \Rightarrow \lambda = 2 \text{ or } -1$$

77) (b) $(\ln x) \int_{2x^2}^{x^3} f(t)\,dt = x^2 - 2x + 5$

diff. w.r. to x

$$\Rightarrow \frac{1}{x} \int_{2x^2}^{x^3} f(t)\,dt + \ln x \left[3x^2 f(x^3) - 4x f(2x^2)\right] = 2x - 2$$

sub $x = 2$

$$\Rightarrow \frac{1}{2} \int_8^8 f(t)\,dt + \ln 2 \left[12 f(8) - 8 f(8)\right] = 2$$

$$\Rightarrow 0 + (\ln 2)[4 f(8)] = 2$$

$$\Rightarrow f(8) = \frac{1}{2 \ln 2}$$

78) (d) ~~$f(\pi/3)$~~ since $f(\theta) = f(\pi - \theta) \Rightarrow f$ is many one function

ie $f(\pi/3) = f(2\pi/3)$, $f(\pi/4) = f(3\pi/4)$.

$$f(x) = \begin{cases} \sin x - \cos x = \sqrt{2}\sin(x - \pi/4), & 0 \le x \le \pi/2 \\ \sin x + \cos x = \sqrt{2}\sin(x + \pi/4), & \pi/2 \le x \le \pi \end{cases}$$

$0 \le x \le \pi/2 \Rightarrow -\pi/4 \le x - \pi/4 \le \pi/4$

$\Rightarrow -1 \le \sqrt{2}\sin(x - \pi/4) \le 1$

Similarly ~~$x + \pi/4$~~ $\pi/2 \le x \le \pi \Rightarrow 3\pi/4 \le x + \pi/4 \le 5$

$-1 \le \sqrt{2}\sin(x + \pi/4) \le 1$

So range of $f(x)$ is $[-1, 1] \ne$ codomain

$\Rightarrow$ into function.

79) (d) $\lim_{x\to 0} \dfrac{\int_0^x x e^{t^2} dt}{1 + x - e^x} = \lim_{x\to 0} \dfrac{x\int_0^x e^{t^2} dt}{1 + x - e^x}$

$= \lim_{x\to 0} \dfrac{\int_0^x e^{t^2} dt + x \cdot e^{x^2}}{1 - e^x}$ (L.H. rule

$= \lim_{x\to 0} \dfrac{e^{x^2} + e^{x^2} + 2x^2 \cdot e^{x^2}}{-e^x}$ (L.H. rules

$= \dfrac{1 + 1 + 0}{-1} = -2$

80) (d) $\lim_{x\to 0} \dfrac{\int_{-x}^{x} f(t)\,dt}{\int_0^{2x} f(t+4)\,dt} = \lim_{x\to 0} \dfrac{f(x) - [f(-x)](-1)}{2f(2x+4)}$ (L.H. rule

$= \lim_{x\to 0} \dfrac{f(x) + f(-x)}{2f(2x+4)}$

$= \dfrac{2f(0)}{2f(4)} = \dfrac{f(0)}{f(4)}$

81 (a) Slope of normal at $x = 0$ is 3

$\Rightarrow -\frac{1}{f'(0)} = 3 \Rightarrow f'(0) = -\frac{1}{3}$

$$\lim_{x \to 0} \frac{x^2}{f(x^2) - 5f(4x^2) + 4f(7x^2)} = \lim_{x \to 0} \frac{2x}{2xf'(x^2) - 40xf'(4x^2) + 56xf'(7x^2)}$$

$$= \lim_{x \to 0} \frac{1}{f'(x^2) - 20f'(4x^2) + 28f'(7x^2)}$$

$$= \frac{1}{f'(0) - 20f'(0) + 28f'(0)} = \frac{1}{9f'(0)}$$

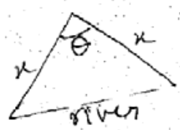
$$= \frac{1}{9 \times (-1/3)} = -\frac{1}{3}$$

82 (d) $p^2 + q^2 = 1$

$\Rightarrow$ let $p = \cos\theta$, $q = \sin\theta$

$p + q = \cos\theta + \sin\theta \Rightarrow$ maximum value $= \sqrt{2}$

83 (b)



Let angle between two sides of fence be θ

$\Rightarrow$ Area $= \frac{1}{2} x \cdot x \cdot \sin\theta = \frac{1}{2} x^2 \sin\theta$

maximum area $= \frac{1}{2} x^2$ (max. value of $\sin\theta = 1$)

84 (b) $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$

Let the point be $a\cos\theta$, $2\sin\theta$

$l^2 = a^2\cos^2\theta + (2\sin\theta + 2)^2$

$= a^2\cos^2\theta + 4\sin^2\theta + 8\sin\theta + 4$

$= a^2\cos^2\theta + 4 - 4\cos^2\theta + 8\sin\theta + 4$

$= (a^2 - 4)\cos^2\theta + 8\sin\theta + 8$

$\frac{dl^2}{d\theta} = (a^2 - 4)(-2\sin\theta\cos\theta) + 8\cos\theta = 0$

$\Rightarrow 2\cos\theta[-(a^2 - 4)\sin\theta + 4] = 0$

$\Rightarrow \cos\theta = 0$ or $\sin\theta = \frac{4}{a^2-4}$

since $4 < a^2 < 8 \Rightarrow 0 < a^2-4 < 4$

$\Rightarrow \sin\theta = \frac{4}{a^2-4} > 1 \Rightarrow$ not possible

so $\cos\theta = 0$

$\Rightarrow \sin\theta = \pm 1$

when $\sin\theta = -1$ then point $(a\cos\theta, 2\sin\theta)$ & $(0,-2)$ coincide

$\Rightarrow \sin\theta = 1$

Hence point is $(0,2)$

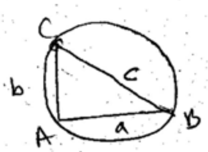
(85) (b) $f'(x) = \frac{1}{3(1+x)^{2/3}} - \frac{1}{3(x-1)^{2/3}} = \frac{(x-1)^{2/3} - (x+1)^{2/3}}{3(x+1)^{2/3}(x-1)^{2/3}}$

$< 0, \ \forall\ x \in [0,1)$

$\Rightarrow f(x)$ is decreasing function

$f_{max} = f(0) = 1^{1/3} - (-1)^{1/3} = 2$

(87) (b)



let $AB = c$ & $AC = b$

$\frac{1}{2}ab = S$

Area of circle $= \pi r^2$

$= \pi\left(\frac{c}{2}\right)^2$

$= \frac{\pi}{4}c^2 = \frac{\pi}{4}(a^2+b^2)$

Since $\frac{a^2+b^2}{2} \geq ab$ (A.M. $\geq$ G.M)

$a^2+b^2 \geq 4ab$

$a^2+b^2 \geq 4\times 2S$

$\Rightarrow$ Area $= \frac{\pi}{4}(a^2+b^2) \geq \frac{\pi}{4}\times 4\times 2S$

$\geq 2\pi S$

(86) (c) slope(m) $= \frac{dy}{dx} = -3x^2 + 6x + 2$

$\frac{dm}{dx} = -6x + 6 = 0 \Rightarrow x = 1$

$m_{max.} = m(1) = -3 + 6 + 2 = 5$

(88) (a) $\lim_{h\to 0} \frac{f(1+h)}{h} = \cancel{\ \ }$ is $\frac{f(1)}{0}$ form

If $f(1)$ is non zero $\Rightarrow$ limit is ∞

but limit is finite (=5) $\Rightarrow f(1) = 0$

$\lim_{h\to 0} \frac{f(1+h)}{h} = \lim_{h\to 0} \frac{f'(1+h)}{1}$ (L-H rule)

$= f'(1) = 5$

(89) (b) $f'(x) = \frac{3}{2}\sqrt{x} + \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$

$f(x)$ is cont. at every $x \geq 0$ & $f'(x)$ is also

also cont. & diff $\forall\ x \in [0, \infty)$

(90) (c) $g'(x^2) = x^3$

$\Rightarrow g'(x) = x^{3/2}$

$\Rightarrow g(x) = \frac{2x^{5/2}}{5} + c$

$g(1) = 1 = \frac{2}{5} + c \Rightarrow c = 3/5$

$g(x) = \frac{2}{5}x^{5/2} + 3/5$

$g(4) = \frac{2}{5} \cdot 4^{5/2} + 3/5 = \frac{2}{5} \times 32 + \frac{3}{5}$

$= \frac{67}{5}$