Mathematics Solutions 1752

Maths - CLASS TEST / 1752

Q1 $f(x) = \begin{cases} x|e^{-x}-2|, & x \le 0 \\ x|e^{x}-2|, & x \ge 0 \end{cases} = \begin{cases} x(e^{-x}-2), & x \le -\ln 2 \\ x(2-e^{-x}), & -\ln 2 \le x \le 0 \\ x(2-e^{x}), & 0 \le x \le \ln 2 \\ x(e^{x}-2), & x \ge \ln 2 \end{cases}$

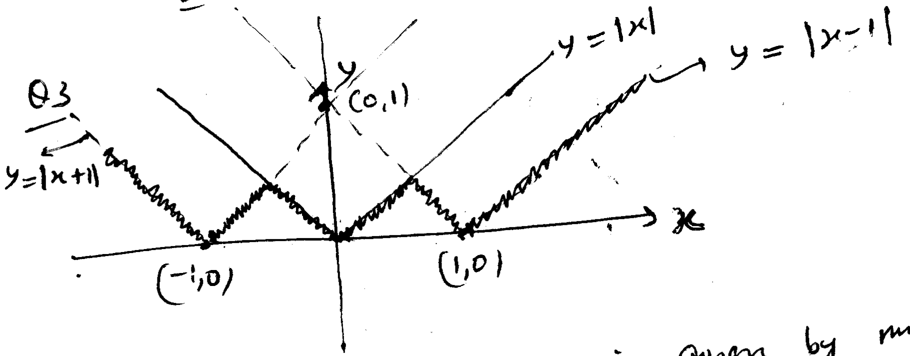
three doubt points $x = -\ln 2, 0, \ln 2$
$f(x)$ is continuous every where but not differentiable at $x = -\ln 2, \ln 2$

Q2 $\lim_{x\to 0} \dfrac{\int_0^x \frac{\ln(1+\sin^2 t)}{t^4+4}dt}{x^3} = \lim_{x\to 0} \dfrac{\frac{\ln(1+\sin^2 x)}{(x^4+4)}}{3x^2}$

$= \lim_{x\to 0} \dfrac{\ln(1+\sin^2 x)}{\sin^2 x} \cdot \dfrac{\sin^2 x}{3x^2(x^4+4)} =$

$= \lim_{x\to 0} \dfrac{1\times 1}{3(x^4+4)} = \dfrac{1}{12}$

Q3



Graph of $y = f(x)$ is given by lines
From graph $f(x)$ turns sharply at five values of x
$x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$ so non differentiable at these values of x.

Q4 $\lim_{x\to a_m} \dfrac{(x-a_1)}{|x-a_1|} \cdot \dfrac{(x-a_2)}{|x-a_2|} \dfrac{(x-a_3)}{|x-a_3|} \cdots \dfrac{(x-a_{m-1})}{|x-a_{m-1}|} \dfrac{(x-a_m)}{|x-a_m|} \dfrac{(x-a_{m+1})}{|x-a_{m+1}|} \cdots \dfrac{}{(x-a_n)}$

L.H.L.

$$\lim_{x\to a_m -} \left(\frac{x-a_1}{x-a_1}\right)\left(\frac{x-a_2}{x-a_2}\right) \cdots \left(\frac{x-a_{m-1}}{x-a_{m-1}}\right)\left(\frac{x-a_m}{-(x-a_m)}\right)\left(\frac{x-a_{m+1}}{-(x-a_{m+1})}\right) \cdots \left(\frac{x-a_n}{-(x-a_n)}\right)$$

$$= 1^{m-1}\cdot(-1)^{n-(m-1)} = (-1)^{n-(m-1)}$$

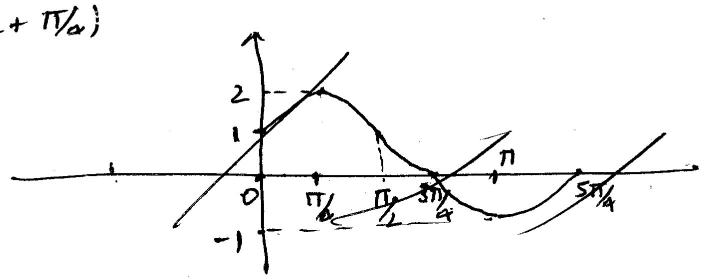
while R.H.L. is $(-1)^{n-m}$

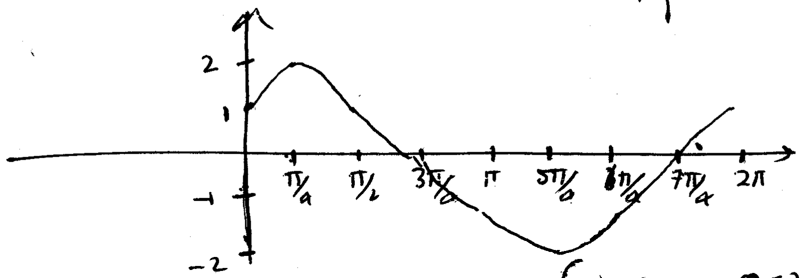
so L.H.L. $\neq$ R.H.L. $\Rightarrow$ limit does not exist.

(5) ~~f (sinx + cosx)~~ $\sin x + \cos x = \sqrt{2}\sin(x+\pi/4)$

It's graph

$y = \sqrt{2}\sin(x+\pi/4)$





so $f(\sin x + \cos x) = \begin{cases} 1, & 0 \le x < 3\pi/4 \\ 0, & x = 3\pi/4 \\ -1, & \frac{3\pi}{4} < x < \frac{7\pi}{4} \\ 0, & x = 7\pi/4 \\ 1 & \frac{7\pi}{4} < x \le 2\pi \end{cases}$

Hence $f\{g(x)\}$ is discont. at $x = 3\pi/4$ & $\frac{7\pi}{4}$

(6) At $x = 1$

L.H.L. $\lim_{x\to 1-} \frac{(x^2-1)(x^2-4)}{[(x-1)(x-2)]} = \lim_{x\to 1-} \frac{\cancel{(x-1)}(x+1)(x-2)(x+2)}{\cancel{(x-1)}(x-2)}$

$= (-1)(3) = -1$

R.H.L.

$$\lim_{x\to 1^+} \frac{(x-1)(x+1)(x-2)(x+2)}{|(x-1)(x-2)|} = \lim_{x\to 1^+} \frac{\cancel{(x-1)}(x+1)(x-2)(x+2)}{-\cancel{(x-1)}(x-2)}$$

$$= \frac{2(-1)(3)}{-(-1)} = -6$$

Hence L.H.L. $\neq$ R.H.L. $\Rightarrow$ dis cont. at $x=1$

Similarly dis cont. at $x=2$.

(7) $$[\cos \pi x] = \begin{cases} 0, & 0\le x\le \frac{1}{2} \\ -1, & \frac{1}{2} < x < 1 \end{cases}$$

Hence $$f(x) = \begin{cases} 0, & 0\le x\le \frac{1}{2} \\ -1, & \frac{1}{2}<x<1 \\ x-1, & 1\le x<2 \end{cases} \Rightarrow$$ discont. at $x=\frac{1}{2}$ & $x=1$ & Hence non diff.

At $x=-1$

L.H.L. $\lim_{x\to -1} [\cos \pi x] = -1$

R.H.L. $\lim_{x\to -1^+} [\cos \pi x] = -1$

so cont. at $x=-1$ & also differentiable $f'(-1)=0$

(8) $$f(x) = \begin{cases} e^x, & x\le 0 \\ 1-x, & 0<x\le 1 \\ x-1, & x\ge 1 \end{cases} \Rightarrow$$ cont. every where but non diff. at $x=0,1$

~~(9)~~ $\cancel{f(x) = \begin{cases} (\pi-x)\frac{\cos x}{\sin x}, & x \end{cases}}$

(9) L.H.L. $$\lim_{x\to\pi^-} \frac{(\pi-x)\cos x}{|\sin x|} = \lim_{x\to\pi^-} \frac{(\pi-x)\cos x}{\sin x}$$

$$= \lim_{x\to\pi^-} \frac{(\pi-x)\cos x}{\sin(\pi-x)} = 1\times\cos\pi = -1$$

R.H.L. $$\lim_{x\to\pi^+} \frac{(\pi-x)\cos x}{-\sin x} = 1$$

Hence dis cont. at $x=\pi$.

(10)
$$f(x) = \begin{cases} -\log_{10} x = -\dfrac{\ln x}{\ln 10}, & 0 < x \le 1 \\ \log_{10} x = \dfrac{\ln x}{\ln 10}, & x \ge 1 \end{cases}$$

Continuous at $x = 1$ but not diff. $f'(1-) = -\dfrac{1}{\ln 10} = -\log_{10} e$

$$f'(1+) = \frac{1}{\ln 10} = \log_{10} e$$

(11)
$$x^{\frac{l+m}{(m-n)(n-l)} + \frac{m+n}{(n-l)(l-m)} + \frac{n+l}{(l-m)(m-n)}}$$

$$x^{\frac{l^2-m^2+m^2-n^2+n^2-l^2}{(m-n)(n-l)(l-m)}} = x^0 = 1$$

so diff. coeff $= 0$

(12)
$$f(x) = \sqrt{x-1} + \sqrt{(x-1)+25-2\times 5\sqrt{x-1}}$$

$$= \sqrt{x-1} + \sqrt{(\sqrt{x-1}-5)^2}$$

$$= \sqrt{x-1} + |\sqrt{x-1}-5|$$

$$= \sqrt{x-1} + 5 - \sqrt{x-1}, \quad 1 < x < 26$$

$$= 5$$

$$f'(x) = 0$$

(13)
$$f(x) = \begin{cases} \frac{1}{2}(-x^2+1), & x < -1 \\ \sin^{-1} x, & -1 \le x \le 1 \\ \frac{1}{2}(x^2+1), & x > 1 \end{cases}$$

dis cont. at $x = -1, 1$ & hence non differentiable

(14)
$$f(x) = \begin{cases} (m-1)\tan \pi x & m-\delta < x < m \\ m \tan \pi x, & m \le x < m+\delta \end{cases}$$

$$f'(m-) = (m-1)(\sec^2 \pi x)\,\pi \Big|_{x=m}$$

$$= (m-1)\pi \sec^2 \pi m = (m-1)\pi$$

$$- (-1)^{2m} (m-1)\pi$$

(15) cont. at $x = 2 \Rightarrow \lim_{x\to 2} \frac{x^2-d}{x-2} = \cancel{dd}$

$\Rightarrow d = 4$ (If d is not 4 then limit is ∞)

so $\lim_{x\to 2} \frac{x^2-4}{x-2} = \mu$

$4 = \mu$

(16) $f(x)$ is continuous at $x = 0$

$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h}$ ($\frac{0}{0}$ form

$= \lim_{h\to 0} \frac{1/h}{e^{1/h^2}} = \lim_{h\to 0} \frac{-1/h^2}{e^{1/h^2}(-2/h^3)}$

$= \lim_{h\to 0} \frac{h}{2e^{1/h^2}} = \frac{0}{\infty} = 0$

(17) $f'(x) = \frac{3}{2}\sqrt{x} - \sqrt{x+1} - \frac{x}{2\sqrt{x+1}}$

$f'(0) = -1$ which is defined so $f(x)$ is differentiable at $x = 0$

(18) $1 \le x \le 2$, $x+1$ is +ve so $sgn(x+1) = 1$

$$f(x) = \begin{cases} x^2 e^{2(1-x)}, & 0 \le x \le 1 \\ a\cos(2x-2) + bx^2, & 1 < x \le 2 \end{cases}$$

For cont. at $x = 1$

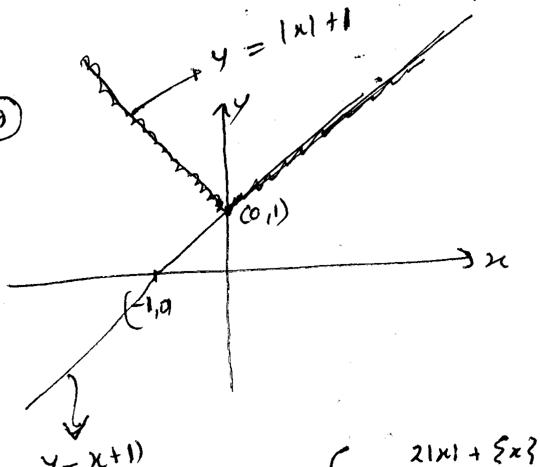
$\Rightarrow 1 \times e^0 = a\cos 0 + b \Rightarrow a + b = 1$

For differentiability.

L.H.D. $f'(1^-) = 2x\, e^{2(1-x)} + x^2 e^{2(1-x)}(-2) \Big|_{x=1}$

$= 2 - 2 = 0$

(19)



$$f(x) = \begin{cases} x+1, & x \le 0 \\ x+1, & x \ge 0 \end{cases}$$

$f(x) = x+1$

$f(x)$ is cont. & diff. $\forall\ x \in R$

(20) $$f(x) = \begin{cases} \dfrac{a^{2|x| + \{x\}} - 1}{2[x] + \{x\}}, & x \ne 0 \\ \ln a, & x = 0 \end{cases}$$

L.H.L. $\displaystyle \lim_{x \to 0^-} \frac{a^{-2x + x - [x]} - 1}{2(-1) + x - [x]} = \lim_{x \to 0^-} \frac{a^{-x+1} - 1}{x - 1}$

$= \dfrac{a - 1}{-1} = 1 - a$

R.H.L. $\displaystyle \lim_{x \to 0^+} \frac{a^{2x + x - [x]} - 1}{\cancel{2\times} 0 + x - [x]} = \lim_{x \to 0^+} \frac{a^{3x} - 1}{x}$

$= 3 \ln a$

so $f(x)$ has irremovable discontinuity.

(21) L.H.L. $\lim_{x \to 0^-} [\tan^2 x] = 0$

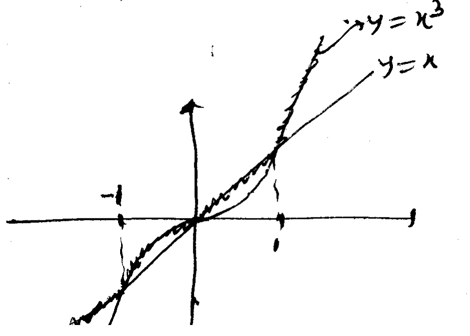
R.H.L. $\lim_{x \to 0^+} [\tan^2 x] = 0$

$\Rightarrow$ continuous at $x = 0$

& diff. at $x = 0$

$f'(0) = 0$

(22)



$$f(x) = \begin{cases} x, & x \le -1 \\ x^3, & -1 \le x \le 0 \\ x, & 0 \le x \le 1 \\ x^3, & x \ge 1 \end{cases}$$

(23) $|x-1| = \begin{cases} -(x-1), & x \le 1 \\ x-1, & x \le 1 \end{cases}$ so $p = -1$

$$\lim_{x \to 1^+} \frac{(x-1)^m}{m \ln(\cos(x-1))} = -1$$

$$\lim_{x \to 1^+} \frac{(x-1)^m}{m \frac{\ln\left(1-2\sin^2\left(\frac{x-1}{2}\right)\right)}{-2\sin^2\left(\frac{x-1}{2}\right)} \left(-2\sin^2\frac{x-1}{2}\right)} = -1$$

$= 1$

$$= \lim_{x \to 1^+} -\frac{1}{2m} \frac{(x-1)^m}{\frac{\sin^2\left(\frac{x-1}{2}\right)}{\left(\frac{x-1}{2}\right)^2} \left(\frac{x-1}{2}\right)^2} = -1$$

$= 1$

$$= \lim_{x \to 1^+} -\frac{1}{2m} \frac{(x-1)^m}{(x-1)^2} \times 4 = -1$$

$$= \lim_{x \to 1^+} -\frac{2}{m} (x-1)^{m-2} = -1$$

$\Rightarrow m = 2$

$= \cancel{\lim_{x \to 1^+} \frac{2m}{2} = -1 \Rightarrow m = -1}$

$\Rightarrow$ $\cancel{2m}$ $\cancel{\frac{2 \times 2}{m} = 1}$ $\quad -\frac{2}{m} = -1$

$\Rightarrow \cancel{m = 4}$ $\quad \Rightarrow m = 2$

(24) $f(x) = \begin{cases} |x|, & x \ne \pm 1 \\ 0, & x = \pm 1 \end{cases}$

$= \begin{cases} -x, & x < -1 \\ 0, & x = -1 \\ -x, & -1 < x < 0 \\ x, & 0 < x < 1 \\ \ldots \end{cases}$ $\Rightarrow$ $f(x)$ is discont. at $x = \pm 1$

$f(x)$ is cont. but ... at $x = 0$

(25) $\lim_{x \to \pi/2^-} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} = \left(\frac{6}{5}\right)^{\frac{\tan(3\pi-)}{\tan(5\pi/2)}}$

$= \left(\frac{6}{5}\right)^{\frac{0}{\infty}} = 1$

$\lim_{x \to \pi/2^+} \left(1 + |\cos x|\right)^{\frac{a}{b}|\tan x|}$

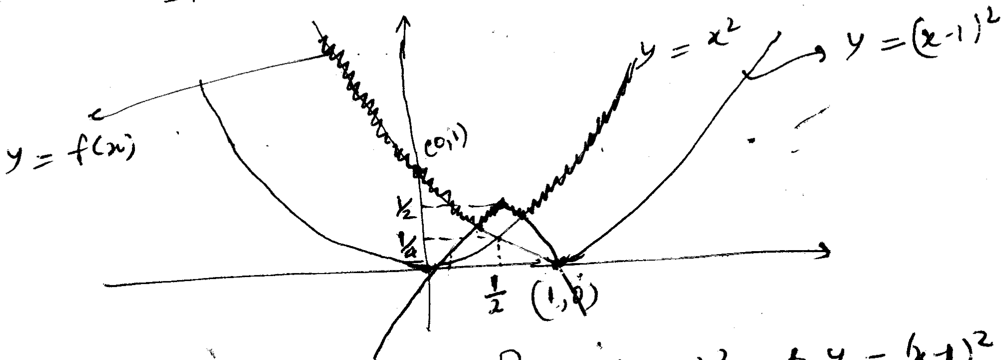
$= e^{\lim_{x \to \pi/2^+} (1 + |\cos x| - 1) \frac{a}{b}|\tan x|} = e^{\lim_{x \to \pi/2^+} \frac{a}{b}|\sin x|}$

$= e^{a/b}$

so $e^{a/b} = 1 = b+2$

$\Rightarrow a = 0$ & $b = -1$

(26)



point of intersection of $y = x^2$ & $y = (x-1)^2$

$x^2 = (x-1)^2 \Rightarrow x = \pm(x-1)$

$x = \frac{1}{2}$

& $y = \frac{1}{4}$

For curve $y = 2x(1-x)$

when $x = 1/2$

$\Rightarrow y = 2 \times \frac{1}{2}(1 - 1/2) = \frac{1}{2}$ is above the both other curves.

From figure $f(x)$ is non diff. at two values of x which are intersection of

$y = x^2$ & $y = 2x(1-x)$

& $y = (1-x)^2$ & $y = 2x(1-x)$

(28) L.H.L.

$\lim_{x\to 0^-} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}$ if $x\to 0^-$ then $e^{1/x} \to e^{-\infty} = 0$

$= \dfrac{0(0+4)}{2-0} = 0$

R.H.L.

$\lim_{x\to 0^+} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}$ if $x\to 0^+ \Rightarrow e^{1/x} \to e^{\infty} \to \infty$

$\lim_{x\to 0^+} \dfrac{x\left(3+\frac{4}{e^{1/x}}\right)}{\frac{2}{e^{1/x}}-1} = \dfrac{0\left(3+0\frac{4}{\infty}\right)}{\frac{2}{\infty}-1} = 0$

$f(0) = 0 \Rightarrow$ cont. at $x=0$

L.H.D.

$f'(0^-) = \lim_{h\to 0^+} \dfrac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0^+} \dfrac{\frac{-h(3e^{-1/h}+4)}{2-e^{-1/h}}-0}{h}$

$= \lim_{h\to 0^+} -\left(\dfrac{3e^{-1/h}+4}{2-e^{-1/h}}\right) = -\left(\dfrac{0+4}{2-0}\right)$

$= -2$

R.H.D.

$f'(0^+) = \lim_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \dfrac{\frac{h(3e^{1/h}+4)}{2-e^{1/h}}-0}{h}$

$= \lim_{h\to 0} \dfrac{3e^{1/h}+4}{2-e^{1/h}} = \lim_{h\to 0} \dfrac{3+\frac{4}{e^{1/h}}}{\frac{2}{e^{1/h}}-1}$

$= \dfrac{3+0}{0-1} = -3$

Hence L.H.D. $\neq$ R.H.D. $\Rightarrow$ not diff. at $x=0$

(29) $f(0) = a = \lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{x^2}$ $\left(\frac{\sin x + x}{2}\right)$ $\left(\frac{\sin x - x}{2}\right)$

$= \lim_{x\to 0} \frac{-2\sin\left(\frac{\sin x + x}{2}\right)\sin\left(\frac{\sin x - x}{2}\right)}{x^2 \left(\frac{\sin x + x}{2}\right)\left(\frac{\sin x - x}{2}\right)}$

$= \lim_{x\to 0} -\frac{1}{2}\left(\frac{(\sin x - x)(\sin x + x)}{x^2}\right)$

$= \lim_{x\to 0} -\frac{1}{2}\left(\frac{\sin^2 x - x^2}{x^2}\right)$

$= \lim_{x\to 0} -\frac{1}{2}\left(\frac{2\sin x\cos x - 2x}{2x}\right)$

$= \lim_{x\to 0} -\frac{1}{2}\left(\frac{\sin 2x - 2x}{2x}\right)$

$= \lim_{x\to 0} -\frac{1}{2}\left[\frac{\sin 2x}{2x} - 1\right] = -\frac{1}{2}[1-1] = 0$

(30) $\lim_{h\to 0} \frac{f(2+h) - f(2+\sin h)}{h \cdot \frac{\sin h}{h} \cdot \frac{\tan h}{h} \times h^2}$

$= \lim_{h\to 0} \frac{[f(2+h) - f(2+\sin h)][2+h-(2+\sin h)]}{h^3 [2+h-(2+\sin x)]}$

$= f'(2)$

(instantaneous rate of change of f at x=2)

$= \lim_{h\to 0} \frac{h - \sin h}{h^3} \cdot f'(2)$

$= \lim_{h\to 0} \frac{1-\cos h}{3h^2} \cdot f'(2) = \lim_{h\to 0} \frac{\sin h}{6h} f'(2)$

$= \frac{f'(2)}{6} = \frac{4}{6} = \frac{2}{3}$