Trigonometric Equations

(1) let angles be $a-d,\ a,\ a+d$

$a-d+a+a+d = 180^\circ \Rightarrow a = 60^\circ$

Smallest $= a-d = (60-d)^\circ = \frac{(60-d)\times\pi}{180}$ radian

Mean $= a = 60^\circ$

$\frac{\frac{(60-d)\pi}{180}}{60} = \frac{\pi}{200} \Rightarrow \frac{(60-d)\pi}{180\times 60} = \frac{\pi}{200}$

$\Rightarrow 60-d = 9\times 6 = 54^\circ$

$\Rightarrow d = 6^\circ$

Greatest angle $= 60+6 = 66^\circ = \frac{66}{180}\pi$ radian

$= \frac{11\pi}{30}$ radians.

(2) $7\left(\frac{1+\cos 2x}{2}\right) + \sin x\cos x - 3 = 0$

$\Rightarrow 7 + 7\cos 2x + 2\sin x\cos x - 6 = 0$

$\Rightarrow 7\cos 2x + \sin 2x = -1$

$\Rightarrow 7\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) + \frac{2\tan x}{1+\tan^2 x} = -1$

$\Rightarrow 7 - 7\tan^2 x + 2\tan x = -1 - \tan^2 x$

$\Rightarrow 6\tan^2 x - 2\tan x - 8 = 0$

$\Rightarrow 3\tan^2 x - \tan x - 4 = 0$

$\Rightarrow 3\tan^2 x + 3\tan x - 4\tan x - 4 = 0$

$\Rightarrow (3\tan x - 4)(\tan x + 1) = 0$

$\Rightarrow \tan x = 4/3$ or -1

$\Rightarrow x = n\pi + \tan^{-1} 4/3$ or $x = n\pi + (-\pi/4)$, $n \in I$

(3) Sum of internal angles of polygon of n sides $= (n-2)\times 180^\circ$.

where n is number of sides or angle

each angle $= \left(\frac{n-2}{n}\right)\times 180 = 150^\circ$

$\frac{n-2}{n} = \frac{\cancel{15}\ 5}{\cancel{18}\ 6} \Rightarrow n = 12$

(4) $2\tan^2 x - 5\sec x = 1$

$\Rightarrow 2(1+\sec^2 x) - 5\sec x = 1$

$\Rightarrow 2\sec^2 x - 5\sec x + 1 = 0$

$\Rightarrow \sec x = \frac{5 \pm \sqrt{25 - 4\times 2}}{2\times 2} = \frac{5 \pm \sqrt{17}}{4}$

$\Rightarrow \sec x = \frac{5+\sqrt{17}}{4}$ or $\frac{5-\sqrt{17}}{4}$ ✗ not possible $\Rightarrow$ less than 1

$\Rightarrow \cos x = \frac{4}{5+\sqrt{17}}$

have solution in 1st & 4th quadrant.

$[0, 6\pi]$ it has 6 solutions (3 complete period & in every period 2 solutions)

One more solution is added when a quadrant is added

$\Rightarrow [0, 6\pi + \pi/2] \Rightarrow [0, 13\pi/2]$

If second & third quadrant are also added then no of solutions still remain same (In 2nd & 3rd quadrant $\cos x$ is negative)

(5) $2\sin 5x \cos 3x = 2\sin 6x \cos 2x$

$\Rightarrow \cancel{\sin 8x} + \sin 2x = \cancel{\sin 8x} + \sin 4x$

$\Rightarrow \sin 4x - \sin 2x = 0$

$\Rightarrow 2\sin 2x \cos 2x - \sin 2x = 0$

$\Rightarrow \sin 2x (2\cos 2x - 1) = 0 \Rightarrow \sin 2x = 0$ or $\cos 2x = 1/2$

$0 \le x \le \pi \Rightarrow 0 \le 2x \le 2\pi$

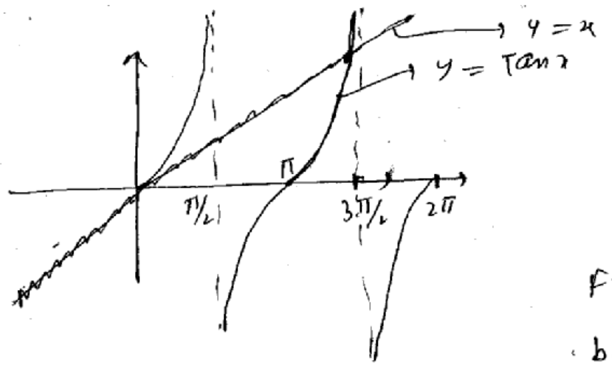
$\sin 2x = 0 \Rightarrow 2x = 0, \pi, 2\pi \Rightarrow$ three solution

$\cos 2x = 1/2 \Rightarrow$ Two solution (one complete cycle of $\cos 2x$)

so Total five solutione.

(6) $\tan x = x$ (This Eqn. is ~~solved g~~ solved graphically)

Draw the graph of $y = \tan x$ & $y = x$ & find pt. of intersection



Since $\tan x > x$, $x \in (0, \pi/2)$
so $(0, \pi/2)$ these curve do not intersect.
From graph they intersect in bt. $(\pi, 3\pi/2)$

(8) $(1-\tan^2\theta)(1+\tan^2\theta) + 2^{\tan^2\theta} = 0$

let $\tan^2\theta = x$

$\Rightarrow 1 - x^2 + 2^x = 0$

$\Rightarrow 2^x = x^2 - 1$ but hit & trial $x = 3$

so $\tan^2\theta = 3 \Rightarrow \tan\theta = \pm\sqrt{3}$

$\Rightarrow \theta = \pi/3, -\pi/3$

(9) $81^{\sin^2 x} + 81^{1-\sin^2 x} = 30$

$81^{\sin^2 x} + \dfrac{81}{81^{\sin^2 x}} = 30$

let $81^{\sin^2 x} = y$

$\Rightarrow y + \dfrac{81}{y} = 30 \Rightarrow y^2 + 81 = 30y$

$\Rightarrow y^2 - 30y + 81 = 0$

$\Rightarrow (y-27)(y-3) = 0$

$\Rightarrow y = 27, 3$

$81^{\sin^2 x} = 3^3, 3$

$\Rightarrow 3^{4\sin^2 x} = 3^3, 3 \Rightarrow 4\sin^2 x = 3, 1$

$\Rightarrow \sin^2 x = \frac{3}{4}, \frac{1}{4}$

$\Rightarrow x = \pi/3, \pi/6$ in $(0, \pi/2)$

(10) $\dfrac{1}{1-\sin\theta} = 4 + 2\sqrt{3}$ (Sum of infinite G.P.)

$\Rightarrow 1 - \sin\theta = \dfrac{1 \cdot (4-2\sqrt{3})}{(4+2\sqrt{3})(4-2\sqrt{3})} = \dfrac{4-2\sqrt{3}}{4} = 1 - \sqrt{3}/2$

$\Rightarrow \sin\theta = \sqrt{3}/2 \Rightarrow \theta = \pi/6$

(11) since $a\sin x + b\cos x$ lies b/t $-\sqrt{a^2+b^2}$ to $\sqrt{a^2+b^2}$,
but $|c| > \sqrt{a^2+b^2}$ $\Rightarrow$ L.H.S. & R.H.S. can not be equal so there does not exist any x for which L.H.S. & R.H.S. are equal

(12) $f(x) = \cos^2 x + \frac{1}{\cos^2 x} \geq 2$

(The sum of magnitude of two numbers which are reciprocal of each other is more than or equal to 2) so (d) is correct.

(13) $\sin x + \sin y = 2$

Since $\sin x, \sin y \leq 1$ but their sum is 2

$\Rightarrow \sin x = 1$ & $\sin y = 1$

$\Rightarrow x = \pi/2$ & $y = \pi/2$ so $x + y = \pi$

(14) $\sin^2 24^\circ - \sin^2 6^\circ = \sin(24+6)\sin(24-6)$

$= \sin 30^\circ \sin 18^\circ$

$= \frac{1}{2}\left(\frac{\sqrt5 - 1}{4}\right) = \frac{\sqrt5 - 1}{8}$

(15) $\sin 2\theta = \sqrt3/2$

$\Rightarrow 2\theta = \pi/3, 2\pi/3$

$\Rightarrow \theta = \pi/6, \pi/3$

$\tan\theta = \frac{1}{\sqrt3}$

$\Rightarrow \theta = \pi/6,$ ~~$7\pi/6$~~

Since both $\sin 2\theta$ & $\tan\theta$ are periodic of period π
so in $[0, \pi]$ common sol. is $\pi/6$

$\Rightarrow$ general sol. $= n\pi + \pi/6$, $n \in I$

alternative (a) $n\pi + 7\pi/6 = n\pi + \pi + \pi/6 = (n+1)\pi + \pi/6$
$= n'\pi + \pi/6$

$\Rightarrow$ (a) is correct.

(16) $\log_{1/2} \sin x > \log_{1/2} \cos x$ $\Rightarrow \sin x, \cos x > 0 \Rightarrow x \in (0, \pi/2)$;
$\sin x < \cos x$ $\Rightarrow x \in (0, \pi/4)$ in first quadrant.

so $x \in (0, \pi/4)$

(17) $\sin 2\pi x = \sqrt{2} \cos \pi x$

$\Rightarrow 2 \sin\pi x \cos\pi x = \sqrt{2} \cos \pi x$

$\Rightarrow \sqrt{2} \cos\pi x \left(\sqrt{2} \sin \pi x - 1\right) = 0$

$\Rightarrow \cos \pi x = 0$, or $\sin \pi x = \frac{1}{\sqrt{2}}$

$\Rightarrow \pi x = \frac{\pi}{2}$ or $\pi x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\Rightarrow x = 1/2$

$\Rightarrow x = \frac{1}{4}, \frac{3}{4}$

greatest value of x is $3/4$

(18) $\sin x = \cos 3x$

$\Rightarrow \cos 3x = \sin x = \cos(\pi/2 - x)$

$\Rightarrow 3x = 2n\pi \pm (\pi/2 - x)$

$+$ sign $\Rightarrow 4x = 2n\pi + \pi/2 \Rightarrow x = \frac{1}{4}\left[2n\pi + \pi/2\right]$ — (1)

$-$ sign $\Rightarrow 2x = 2n\pi - \pi/2 \Rightarrow x = \frac{1}{2}\left[2n\pi - \pi/2\right]$ (2)

sol. are given by (1) & (2)

From (1) $n = 0$, $x = \pi/8$
$n = 1$, $x = 5\pi/8$
$n = 2$, $x = \frac{9\pi}{8} > \pi$ ✗

From (2)
$n = 1$, $x = \frac{3\pi}{4}$
$n = 2$, $x = \frac{7\pi}{4} > \pi$ ✗

So total 3 sol. in $[0, \pi]$

(19) $\sin^2\theta - 2\sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \dfrac{2 \pm \sqrt{4 - 4 \times 1(-1)}}{2} = 1 \pm \sqrt{2}$

$\sin\theta \neq 1 + \sqrt{2} \Rightarrow \sin\theta = 1 - \sqrt{2} \Rightarrow \theta$ is in 3rd & 4th quadrant

($\sin\theta$ is $-$ve)

so 4 sol. in $[0, 4\pi] \Rightarrow n = 4$

(20) $y = \frac{1}{2} = \sin x$

$\Rightarrow \sin x = \frac{1}{2}$ in $[-2\pi, 2\pi]$ $\Rightarrow$ Total 4 solution

(21) $\tan^2\theta + \frac{1}{\cos 2\theta} = 1$

$\Rightarrow \tan^2\theta + \frac{(1+\tan^2\theta)}{1-\tan^2\theta} = 1$

$\Rightarrow \tan^2\theta(1-\tan^2\theta) + 1 + \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow \cancel{\tan^2\theta(1-\tan^2\theta) + \tan^2\theta = 0}$

$\Rightarrow \cancel{\tan^2\theta(2-\tan^2\theta) = 0}$

$\Rightarrow \cancel{\tan^2\theta = 0 \text{ or } 2}$

$\Rightarrow \tan^2\theta(1-\tan^2\theta) + 2\tan^2\theta = 0$

$\Rightarrow \tan^2\theta(3-\tan^2\theta) = 0$

$\Rightarrow \tan^2\theta = 0$ or $\tan^2\theta = 3$

$\Rightarrow \theta = n\pi$ or $\theta = n\pi \pm \pi/3$

(22)

$1-\cos^2\theta + 3\cos\theta = 3 \Rightarrow \cos^2\theta - 3\cos\theta + 2 = 0$

$\Rightarrow \cos\theta = 1, 2$ ✗

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 2n\pi + 0$

Only one of $\theta \in [-\pi, \pi]$ is 0

so only one solution.

(23)

$\sec\theta + \tan\theta = \sqrt{3}$

$\Rightarrow \frac{1+\sin\theta}{\cos\theta} = \sqrt{3} \Rightarrow \frac{1 - \cos(\pi/2+\theta)}{\sin(\pi/2+\theta)} = \sqrt{3}$

$\Rightarrow \frac{2\sin^2\left(\frac{\pi/2+\theta}{2}\right)}{2\sin\left(\frac{\pi/2+\theta}{2}\right)\cos\left(\frac{\pi/2+\theta}{2}\right)} = \sqrt{3}$

$\Rightarrow \tan(\theta/2 + \pi/4) = \sqrt{3}$

$\Rightarrow \theta/2 + \pi/4 = n\pi + \pi/3$

$\theta = 2n\pi + \pi/6$

n	θ
0	$11\pi/6$
2	$23\pi/6 > 3\pi$ so not required

$\Rightarrow \theta = 11\pi/6$ is the only one sol.

(23)

n	θ
0	$\pi/6$
1	$13\pi/6$
2	$25\pi/6 > 3\pi$

so not required $\Rightarrow$ only two sol. in given interval.

24. $3\cos 2x - 10\cos x + 7 = 0$

$3(2\cos^2 x - 1) - 10\cos x + 7 = 0$

$\Rightarrow 6\cos^2 x - 10\cos x + 4 = 0$

$\Rightarrow 6\cos^2 x - 6\cos x - 4\cos x + 4 = 0$

$\Rightarrow (6\cos x - 4)(\cos x - 1) = 0$

$\Rightarrow \cos x = 2/3, 1$

$\cos x = 1$ have sol $x = 0, 2\pi, 4\pi$

$\cos x = 2/3$ have sol. in 1st & 4th quadrant in one complete sol.

so have 5 sol. in $[0, 5\pi]$

$\Rightarrow$ Total 8 solutions

(25) $\tan\theta + \tan\left(\frac{3\pi}{4} + \theta\right) = 2$

$\Rightarrow \tan\theta + \frac{\tan 3\pi/4 + \tan\theta}{1 - \tan 3\pi/4 \tan\theta} = 2$

$\Rightarrow \tan\theta + \frac{-1 + \tan\theta}{1 + \tan\theta} = 2$

$\Rightarrow \tan\theta(1 + \tan\theta) - 1 + \tan\theta = 2(1 + \tan\theta)$

$\Rightarrow \tan^2\theta + \cancel{2\tan\theta} - 3 = 0$

$\Rightarrow \cancel{\tan\theta = -3, 1}$ $\tan^2\theta = 3 = \tan^2 \pi/3$

$\Rightarrow \theta = n\pi \pm \pi/3$

26 $\cos 2x - a\sin x = 2a-7$

$\Rightarrow 1-2\sin^2 x - a\sin x = 2a-7$

$\Rightarrow 2\sin^2 x + a\sin x + 2a-8 = 0$

$\Rightarrow \sin x = \dfrac{-a \pm \sqrt{a^2-4\times 2(2a-8)}}{2\times 2}$

$= \dfrac{-a \pm (a-8)}{4} = -2,\ \dfrac{8-2a}{4}$

Since $\sin x \neq -2$

$\Rightarrow \sin x = \dfrac{8-2a}{4}$ for sol: $-1 \le \dfrac{8-2a}{4} \le 1$

$-1 \le \dfrac{4-a}{2} \le 1$

$-2 \le 4-a \le 2$

$-6 \le -a \le -2$

$\boxed{2 \le a \le 6}$

27 $\cos(a\sin x) = \sin(a\cos x)$

$\Rightarrow \cos(a\sin x) = \cos(\pi/2 - a\cos x)$

$\Rightarrow a\sin x = 2n\pi \pm (\pi/2 - a\cos x)$

$\Rightarrow a\sin x \pm a\cos x = 2n\pi \pm \pi/2$

$\Rightarrow a(\sin x + \cos x) = 2n\pi \pm \pi/2$

For smallest magnitude of a R.H.S. should be least & L.H.S. greatest.

$\Rightarrow a\cdot\sqrt{2} \geq \pi/2$

$\Rightarrow a \geq \dfrac{\pi}{2\sqrt{2}} \approx 1.2$

$\Rightarrow$ ~~least~~ least +ive integral value of a is 2

28 $D \geq 0 \Rightarrow \cos^2 b - 4(\cos b - 1)\sin b \ge 0$

$\Rightarrow \cos^2 b + 4(1-\cos b)\sin b \ge 0$

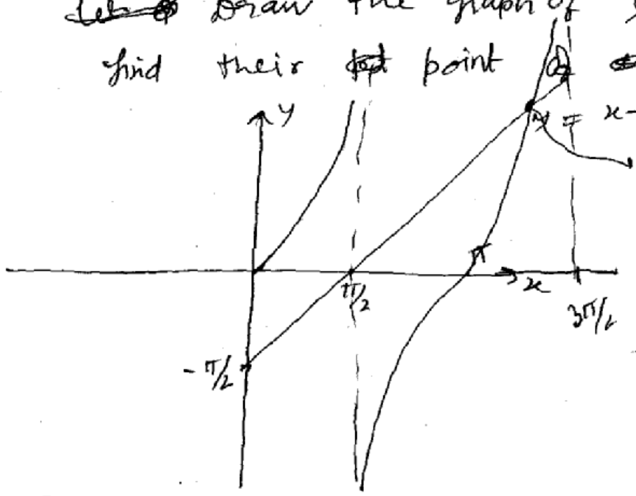
$\cos^2 b$ & $1-\cos b \geq 0$ so from alternative

$b \in (0,\pi)$, $\sin b > 0$ so all the terms are

+ive $\in (0, \pi)$

(29) $x - \pi/2 = 2\tan x$

~~Let~~ Draw the graph of $y = x - \pi/2$ & $y = \tan x$ & find their point of intersection.



point of intersection

From graph these curves intersects only at one point

(30) $\cos x - \sin x \geq 1$

$\Rightarrow \frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}}\sin x \geq \frac{1}{\sqrt{2}}$

$\Rightarrow \cos x \cos \pi/4 - \sin x \sin \pi/4 \geq \frac{1}{\sqrt{2}}$

$\Rightarrow \cos(x + \pi/4) \geq \frac{1}{\sqrt{2}}$

Since $\cos\theta \geq \frac{1}{\sqrt{2}} \Rightarrow$ ~~$\theta \leq \pi/4$ or $\theta \geq$~~

$\theta \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{7\pi}{4}, \frac{9\pi}{4}\right]$

so $\cos(x + \pi/4) \geq \frac{1}{\sqrt{2}}$

$\Rightarrow (x + \pi/4) \in \left[0, \pi/4\right] \cup \left[\frac{7\pi}{4}, \frac{9\pi}{4}\right]$

$\Rightarrow x \in \left[-\pi/4, 0\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

(31) $3\sin x + 4\tan x = 5$

$\frac{6\tan x/2}{1 + \tan^2 x/2} + \frac{4(1 - \tan^2 x/2)}{1 + \tan^2 x/2} = 5$

$6\tan x/2 + 4 - 4\tan^2 x/2 = 5 + 5\tan^2 x/2$

$9\tan^2 x/2 - 6\tan x/2 + 1 = 0$

$(3\tan x/2 - 1)^2 = 0$

$\Rightarrow \tan x/2 = 1/3$

(32) $\cos 3x = -\sin 5x = \cos(\pi/2 + 5x)$

$\Rightarrow$ ~~3x = 2n~~ $\cos(5x + \pi/2) = \cos 3x$

$\Rightarrow 5x + \pi/2 = 2n\pi \pm 3x$

+ sign $\Rightarrow 2x = 2n\pi - \pi/2 \Rightarrow x = n\pi - \pi/4$

− sign $\Rightarrow 8x = 2n\pi - \pi/2 \Rightarrow x = \frac{1}{4}(n\pi - \pi/4)$

Least +ive value is obtained at $n = 1$ in

2nd case $= \frac{1}{8}(\pi - \pi/4) = \frac{1}{4} \times \frac{3\pi}{4}$

$= \frac{3\pi}{16}$

(33) $\sin 3\alpha = 4\sin(x+\alpha)\sin(x-\alpha)\sin\alpha$

$\Rightarrow \sin 3\alpha = 4[\sin^2 x - \sin^2\alpha]\sin\alpha$

$\Rightarrow 3\sin\alpha - \cancel{4\sin^3\alpha} = 4[\sin^2 x \sin\alpha - \cancel{\sin^3\alpha}]$

$\Rightarrow 3\sin\alpha = 4\sin^2 x \sin\alpha$

$\Rightarrow \sin^2 x = \frac{3}{4}$

$\Rightarrow x = n\pi \pm \pi/3 \Rightarrow$ have one solution in each quadrant.

(34) $4\sin\theta\cos\theta - 2\cos\theta - 2\sqrt{3}\sin\theta + \sqrt{3} = 0$

$2\cos\theta(2\sin\theta - 1) - \sqrt{3}(2\sin\theta - 1) = 0$

$(2\sin\theta - 1)(2\cos\theta - \sqrt{3}) = 0$

$\Rightarrow \sin\theta = 1/2$ or $\cos\theta = \sqrt{3}/2$

$\Rightarrow \theta = \pi/6, 5\pi/6$ or $\theta = \pi/6, 11\pi/6$

$\Rightarrow \theta = \pi/6, 5\pi/6, 11\pi/6$

(35) $\sin x \tan 4x = \cos x$

$\Rightarrow$ ~~sin x sin 4x~~ $\Rightarrow \tan x \tan 4x = 1$

$\Rightarrow \tan 4x = \cot x$

$\Rightarrow \tan 4x = \tan(\pi/2 - x)$

$4x = n\pi + \pi/2 - x$

$x = \frac{1}{5}[n\pi + \pi/2] = \frac{1}{5}[n\pi + \pi/2]$

n	x
0	$\frac{\pi}{10}$
1	$\frac{3\pi}{10}$
2	$\frac{5\pi}{10}$
3	$\frac{7\pi}{10}$
4	$\frac{9\pi}{10}$
5	$\frac{11\pi}{10} > \pi$ X

$x = (2n+1)\frac{\pi}{10}$

For $x \in (0, \pi)$, $n = 0, 1, 2, 3, 4$

$\boxed{\Rightarrow \text{ 5 solutions}}$

Alternative. $\rightarrow$ $2\sin x \sin 4x = 2\cos x \cos 4x$

$\cos 3x - \cos 5x = \cos 5x + \cos 3x$

$\Rightarrow \cos 5x = 0$

$5x = (2n+1)\pi/2$

$\Rightarrow x = (2n+1)\frac{\pi}{10}$

(36) $\tan\theta = -1$ & $\cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ | $\theta = \pi/4, \frac{7\pi}{4}$

common solution in $[0, 2\pi]$ is $\frac{7\pi}{4}$

$\Rightarrow$ General sol. $= 2n\pi + \frac{7\pi}{4}$

(37) $2\sin 3x \sin 2x = 2\sin 3x \sin x$

$\Rightarrow \sin 3x = 0$ or $\sin 2x = \sin x$

$\Rightarrow 2\sin x\cos x - \sin x = 0$

$\Rightarrow 3x = n\pi$

$\Rightarrow x = n\pi/3$

$\Rightarrow \sin x(2\cos x - 1) = 0$

$\Rightarrow \sin x = 0$ or $\cos x = 1/2$

$\Rightarrow x = n\pi$, or $x = 2n\pi \pm \pi/3$

Since $x = n\pi/3$ already contains $x = n\pi$ & $2n\pi \pm \pi/3$.

$\Rightarrow$ general sol. $x = \frac{n\pi}{3}$

(38) $A + B + C = 180$ & $2B = A + C$

$\Rightarrow B + 2B = 180 \Rightarrow B = 60^\circ = \pi/3$ rad.

$\sin(2A+B) = \frac{1}{2}$

$\Rightarrow 2A + B = \pi/6$ or $\frac{5\pi}{6}$

since $B = \pi/3 \Rightarrow 2A + B = \frac{5\pi}{6}$

$\Rightarrow 2A + \pi/3 = \frac{5\pi}{6}$

$2A = \frac{3\pi}{6}$

$A = \pi/4$

& $C = \pi - \pi/3 - \pi/4 = \frac{5\pi}{12}$

(39) $2\cos 3\theta = 2\cos 2\theta - 1$

$\Rightarrow 2[4\cos^3\theta - 3\cos\theta] - 2[2\cos^2\theta - 1] + 1 = 0$

$\Rightarrow 8\cos^3\theta - 4\cos^2\theta - 6\cos\theta + 3 = 0$

$\Rightarrow 4\cos^2\theta(2\cos\theta - 1) - 3(2\cos\theta - 1) = 0$

$(2\cos\theta - 1)(4\cos^2\theta - 3) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos^2\theta = \frac{3}{4}$

$\Rightarrow \theta = 2n\pi \pm \pi/3$ or $\theta = \cancel{2} n\pi \pm \pi/6$

(40) $\tan x + \frac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x} + \frac{\tan x - \sqrt{3}}{1 - (-\sqrt{3})\tan x} = 3$

$\Rightarrow \tan x + \frac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x} + \frac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x} = 3$

$$\Rightarrow \frac{\tan x(1-\sqrt{3}\tan x)(1+\sqrt{3}\tan x) + (\tan x + \sqrt{3})(1+\sqrt{3}\tan x) + (\tan x - \sqrt{3})(1 - \sqrt{3}\tan x)}{(1-\sqrt{3}\tan x)(1+\sqrt{3}\tan x)} = 3$$

$$\Rightarrow \frac{\tan x(1 - 3\tan^2 x) + \cancel{\sqrt{3}\tan^2 x} + 4\tan x + \cancel{\sqrt{3}} - \cancel{\sqrt{3}\tan^2 x} + 4\tan x - \cancel{\sqrt{3}}}{1 - 3\tan^2 x} = 3$$

$\Rightarrow \frac{9\tan x - 3\tan^3 x}{1 - 3\tan^2 x} = 3 \Rightarrow 3\tan 3x = 3$

$\Rightarrow \tan 3x = 1 = \tan \pi/4$

$\Rightarrow 3x = n\pi + \pi/4 \Rightarrow x = \frac{1}{3}\left[n\pi + \pi/4\right]$

(41) $\tan^2\alpha + 2\sqrt{3}\tan\alpha - 1 = 0$

$\Rightarrow \tan\alpha = \dfrac{-2\sqrt{3} \pm \sqrt{12 - 4(-1)}}{2} = -\sqrt{3} \pm 2$

$\tan\alpha = 2 - \sqrt{3}$ or $-(2+\sqrt{3})$

$\tan\alpha = \tan\frac{\pi}{12}$ or $\tan\left(\frac{105}{180} \times \pi\right) = \tan\left(\frac{7}{12}\pi\right)$

$\alpha = n\pi + \frac{\pi}{12}$ or $\alpha = n\pi + \frac{7\pi}{12}$

$= (12n+1)\frac{\pi}{12}$ or $\alpha = (12n+7)\frac{\pi}{12}$

Both the solutions in combined form can be written as $(6n+1)\frac{\pi}{12}$

Passage.

(42) $\sec\theta + \operatorname{cosec}\theta = 2\sqrt{2}$

$\frac{1}{\cos\theta} + \frac{1}{\sin\theta} = 2\sqrt{2}$

$\dfrac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = 2\sqrt{2}$ — (1)

Let $\sin\theta + \cos\theta = t$,

squaring $\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = t^2$

$\Rightarrow \sin\theta\cos\theta = \frac{t^2-1}{2}$

sub. in (1)

$\frac{2t}{t^2-1} = 2\sqrt{2} \Rightarrow t^2 - 1 = \frac{1}{\sqrt{2}}t$

$\Rightarrow t^2 - \frac{1}{\sqrt{2}}t - 1 = 0$

$\Rightarrow t = \dfrac{\frac{1}{\sqrt{2}} \pm \sqrt{\frac{1}{2} - 4(-1)}}{2}$

$t = \dfrac{\frac{1}{\sqrt{2}} \pm \frac{3}{\sqrt{2}}}{2} = \sqrt{2}, -\frac{1}{\sqrt{2}}$

$\Rightarrow \sin\theta + \cos\theta = \sqrt{2} \text{ or } -\frac{1}{\sqrt{2}}$

$\Rightarrow \sqrt{2}\sin(\theta + \pi/4) = \sqrt{2} \text{ or } -\frac{1}{\sqrt{2}}$

$\Rightarrow \sin(\theta + \pi/4) = 1, -\frac{1}{2}$

$\theta \in (0, 2\pi)$ $\quad \sin(\theta + \frac{\pi}{4}) = 1 \Rightarrow \theta + \frac{\pi}{4} = \pi/2 \Rightarrow \theta = \pi/4$

$\sin(\theta + \pi/4) = -1/2 \Rightarrow \theta + \pi/4 = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\Rightarrow \theta = \frac{7\pi}{6} - \pi/4, \frac{11\pi}{6} - \frac{\pi}{4}$

$= \frac{11\pi}{12}, \frac{19\pi}{12}$

$\Rightarrow$ ~~3 solution~~ 3 Solutions.

43/44 $\quad \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = c^2$

let $\sin\theta + \cos\theta = t \Rightarrow \sin\theta\cos\theta = \frac{t^2-1}{2}$

$\Rightarrow \frac{2t}{t^2-1} = c^2$

$\Rightarrow t^2 - 1 = \frac{2}{c^2} t$

$\Rightarrow t^2 - \frac{2}{c^2} t - 1 = 0$

$\Rightarrow t = \frac{\frac{2}{c^2} \pm \sqrt{\frac{4}{c^2} + 4}}{2} = \cancel{\frac{2 \pm 2\sqrt{1+c^2}}{2c^2} = \frac{1 \pm \sqrt{1+c^2}}{c^2}}$

$= \frac{1 \pm \sqrt{1+c^2}}{c} = \frac{1+\sqrt{1+c^2}}{c}, \frac{1-\sqrt{1+c^2}}{c}$

$= \frac{1}{c} + \sqrt{\frac{1}{c^2}+1}, \frac{(1-\sqrt{1+c^2})(1+\sqrt{1+c^2})}{c(1+\sqrt{1+c^2})}$

$= \frac{1}{c} + \sqrt{\frac{1}{c^2}+1}, \frac{-c}{1+\sqrt{1+c^2}}$

If $c^2 > 8 \Rightarrow \frac{1}{c^2} < \frac{1}{8}$

so $\left|\frac{1}{c} + \sqrt{\frac{1}{c^2}+1}\right| < \left|\frac{1}{2\sqrt{2}} + \sqrt{\frac{1}{8}+1}\right| < \left|\frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}\right) < \sqrt{2}$

so ~~if $\sin x + \cos x = \frac{1}{c} + \sqrt{\frac{1}{c^2}+1}$~~

similary $\quad -\frac{c}{1+\sqrt{1+c^2}} = -\left[\frac{1}{\frac{1}{c} + \sqrt{\frac{1}{c^2}+1}}\right]$

If $\left| \frac{1}{c} + \sqrt{\frac{1}{c^2}+1} \right| < \sqrt{2}$

$\Rightarrow \left| \frac{1}{\frac{1}{c} + \sqrt{\frac{1}{c^2}+1}} \right| > \frac{1}{\sqrt{2}}$

$\Rightarrow \left| - \frac{1}{\frac{1}{c} + \sqrt{\frac{1}{c^2}+1}} \right| > \frac{1}{\sqrt{2}}$