Maths Solutions

Phase Test - 5 (TYRP – 17)
Test Code. 1747

(61) $R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} \cos x + 2\sin x & \cos x + 2\sin x & \cos x + 2\sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$$

$$\Rightarrow (\cos x + 2\sin x)\begin{vmatrix} 1 & 1 & 1 \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$)

$$\Rightarrow (\cos x + 2\sin x)\begin{vmatrix} 1 & 0 & 0 \\ \sin x & \cos x - \sin x & 0 \\ \sin x & 0 & \cos x - \sin x \end{vmatrix} = 0$$

$\Rightarrow (\cos x + 2\sin x)(\cos x - \sin x)^2 = 0$

$\Rightarrow \cos x + 2\sin x = 0$ or $\cos x = \sin x$

$\Rightarrow \tan x = -\frac{1}{2}$ or $\tan x = 1$

if $x \in [-\pi/4, \pi/4]$ $\tan x$ is increasing from $[-1, 1]$ so there are exactly one x for each of $\tan x = -\frac{1}{2}$ & $\tan x = 1$ so total no. of two solutions

(62) let $x = \frac{1}{y}$ so $y \to 0^+$

$\lim_{y \to 0^+} \sqrt{\frac{1}{y^2} + \frac{1}{y} + 1} - \frac{a}{y} + b = 1$

$\lim_{y \to 0^+} \frac{\sqrt{1 + y + y^2}}{|y|} - \frac{a}{y} + b = 1$

$\lim_{y \to 0^+} \frac{\sqrt{1 + y + y^2} - a + by}{y} = 1$

Direct sub. $\Rightarrow \frac{1-a}{0}$ for limit to be 1 $\Rightarrow a = 1$
sub. a in above

$\lim_{y \to 0^+} \frac{\sqrt{1 + y + y^2} - 1 + by}{y}$ ($\frac{0}{0}$ form)

L.H. rule $\lim_{y \to 0^+} \frac{\frac{(1+2y)}{2\sqrt{1+y+y^2}} + b}{1} = \frac{1}{2} + b = 1$

$\Rightarrow b = \frac{1}{2}$

(63) $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{5}{7}\right)+\tan\left[\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{5}{7}\right)\right]$

$\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{5}{7}\right)+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{5}{7}\right]$

Now sub. $\cos^{-1}5/7=\theta$ &

$\Rightarrow \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$\Rightarrow \frac{1+\tan\theta/2}{1-\tan\theta/2}+\frac{1-\tan\theta/2}{1+\tan\theta/2}=\frac{2(1+\tan^2\theta/2)}{1-\tan^2\theta/2}$

$=\frac{2}{\cos\theta}=\frac{2\times7}{5}$

(64) $B^2=\begin{bmatrix}4&1\\-9&-2\end{bmatrix}\begin{bmatrix}4&1\\-9&-2\end{bmatrix}=\begin{bmatrix}7&2\\-18&-5\end{bmatrix}$

$B^3=\begin{bmatrix}4&1\\-9&-2\end{bmatrix}\begin{bmatrix}7&2\\-18&-5\end{bmatrix}=\begin{bmatrix}10&3\\-27&-8\end{bmatrix}$

$B^4=\begin{bmatrix}10&3\\-27&-8\end{bmatrix}\begin{bmatrix}4&1\\-9&-2\end{bmatrix}=\begin{bmatrix}13&4\\-36&-11\end{bmatrix}$

From the above patten we observe
the terms of B^{100}
a_{11} is 100th term of series 4, 7, 10, 13 – – ie 301
a_{12} is 100th term of 1, 2, 3 – – ie 100
a_{21} is 100th terms of –9, –18, –27 – – ie –900
a_{22} is 100th term of –2, –5, –8, –11 – – ie –299

(65) Apply L.H. rule
$\Rightarrow \lim_{x\to2}\frac{\frac{1}{3}(60+x^2)^{-2/3}(2x)}{\cos(x-2)}=\frac{\frac{1}{3}(64)^{-2/3}\times4}{1}=\frac{1}{12}$

(66) $\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2\times3^n}{3+3^{2n}}\right)=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2\times3^n/3}{1+3^{2n}/3}\right)$

$=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{3^n-3^{n-1}}{1+3^n\cdot3^{n-1}}\right)=\sum_{n=1}^{\infty}$

$=\sum^{\infty}\left(\tan^{-1}3^n-\tan^{-1}3^{n-1}\right)$

(67) $2f(x) + f(1-x) = x$ — (1)

replace x by $1-x$

$\Rightarrow 2f(1-x) + f(x) = 1-x$ — (2)

(2) × 2 − (1) $\Rightarrow 3f(x) = 2x - 1 + x$

$\Rightarrow f(x) = x - 1/3$

(68) $f(x) = \dfrac{1+x^2+x^4}{(1+x+x^2)^2} = \dfrac{(1+x+x^2)(1-x+x^2)}{(1+x+x^2)^2}$

$= \dfrac{1-x+x^2}{1+x+x^2}$

Range of $f(x)$ is $\left[\frac{1}{3}, 3\right]$ & $f(x)$ is both increasing & decreasing in the domain so Into & many one.

(69) $\sin\left[\pi/6 + \sin^{-1}(0.4)\right] \sin\left[\pi/6 - \sin^{-1}(0.4)\right]$

$= \sin^2 \pi/6 - \sin^2 \sin^{-1}(0.4) = \frac{1}{4} - (0.4)^2$

(70) $n(A \times B) = 8$

so Number of subsets having three or more elements are

$= {}^8C_3 + {}^8C_4 + {}^8C_5 + \ldots {}^8C_8$ (${}^8C_4$ → no. of subsets having 4 elements)

$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2$

(71) $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\pi/2$

$\Rightarrow \pi/2 + \sin^{-1} 6x = -\sin^{-1} 6\sqrt{3}x$

$\Rightarrow \sin(\pi/2 + \sin^{-1} 6x) = -6\sqrt{3}x$

$\Rightarrow \cos \sin^{-1} 6x = -6\sqrt{3}x$

$\Rightarrow \sqrt{1-36x^2} = -6\sqrt{3}x$

$\Rightarrow 1 - 36x^2 = 108x^2 \Rightarrow 144x^2 = 1$

$x = \pm\frac{1}{12}$

only $x = -\frac{1}{12}$ satisfies

(72) $\lim_{x\to 0} \frac{1}{x^2}\begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$

$= \lim_{x\to 0} \frac{1}{x^2}\cdot x \begin{vmatrix} \sin x & \cos x & \tan x \\ x^2 & x & 1 \\ 2x & 1 & 1 \end{vmatrix}$

$= \lim_{x\to 0} \begin{vmatrix} \frac{\sin x}{x} & \cos x & \tan x \\ x & x & 1 \\ 2 & 1 & 1 \end{vmatrix}$ (dividing first colum by x)

$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -1 - 1(-2) = 1$

(73) let matrices are A, B, C in order
then we can say $A^T = C$ or $C^T = A$
& B is symmetric matrix.
so given is $(ABA^T)^2 = ABA^TABA^T$ so the matrix is also symmetric matrix

$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is also symmetric

$\Rightarrow a_2 = b_1,\ a_3 = c_1,\ b_3 = c_2$

$|a_2 - b_1| + |a_3 - c_1| + |b_3 - c_2| = 0$

(74) Suppose first term of A.P. $= a$ & common diff. $= d$

$\Rightarrow \begin{vmatrix} e^{a+2d} & e^{a+5d} & e^{a+7d} \\ e^{a+3d} & e^{a+6d} & e^{a+8d} \\ e^{a+4d} & e^{a+6d} & e^{a+9d} \end{vmatrix} =$

$= e^{a+2d}\, e^{a+5d}\, e^{a+7d} \begin{vmatrix} 1 & 1 & 1 \\ e^d & e^d & e^d \\ e^{2d} & e^{2d} & e^{2d} \end{vmatrix} = 0$

(75) $A - B = \tan^{-1}\left(\frac{x\sqrt{3}}{2k-x}\right) - \tan^{-1}\left(\frac{2x-k}{k\sqrt{3}}\right)$

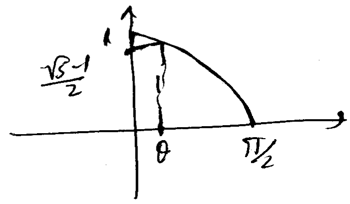
$$= \tan^{-1}\left[\frac{\frac{x\sqrt{3}}{2k-x} - \left(\frac{2x-k}{k\sqrt{3}}\right)}{1 + \left(\frac{x\sqrt{3}}{2k-x}\right)\cdot\left(\frac{2x-k}{k\sqrt{3}}\right)}\right]$$

$$= \tan^{-1}\left[\frac{3xk - (2x-k)(2k-x)}{k\sqrt{3}(2k-x) + x\sqrt{3}(2x-k)}\right]$$

$$= \tan^{-1}\left[\frac{-kx + 2k^2 + 2x^2}{2\sqrt{3}k^2 - \sqrt{3}kx + 2\sqrt{3}x^2 - \sqrt{3}kx}\right]$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$$

(76) $\alpha \in \left(\cos^{-1}\frac{\sqrt{5}-1}{2}, \pi/2\right)$



$\frac{\sqrt{5}-1}{2} \approx \frac{2.2-1}{2} \approx 0.6$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx \frac{1.4}{2} \approx 0.7$

$\cos^{-1}\frac{1}{\sqrt{2}} = \pi/4$ so $\cos^{-1}\frac{\sqrt{5}-1}{2} > \pi/4$

since $\alpha > \pi/4$ & is in the first quadrant

so $\sin\alpha > \cos\alpha$

$\tan\alpha > \sin\alpha$ (because $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, $\cos\alpha < 1$

so $\tan\alpha > \sin\alpha$

so $\tan\alpha > \sin\alpha > \cos\alpha$

if $\alpha \in (\pi/4, \pi/2)$ so $\tan\alpha > \cot\alpha$

Now we need to compare $\cot\alpha$ & $\sin\alpha$

for equality $\cot\alpha = \sin\alpha$

$\frac{\cos\alpha}{\sin\alpha} = \sin\alpha$

$\Rightarrow \cos\alpha = \sin^2\alpha = 1 - \cos^2\alpha$

$\cos\alpha = \frac{-1 \pm \sqrt{5}}{2}$ ~~is~~ . so for first quadrant

$\cos\alpha = \frac{-1+\sqrt{5}}{2}$

$\Rightarrow$ when $\cos\alpha = \frac{\sqrt{5}-1}{2}$ then $\sin\alpha$ & $\cot\alpha$ are equal.

~~a~~ as ~~cb~~ α increases from this value $\sin\alpha$ increases & $\cot\alpha$ decreases ~~so~~ so in given interval

$\sin\alpha > \cot\alpha$

$\Rightarrow$ correct order $\cos\alpha < \cot\alpha < \sin\alpha < \tan\alpha$

(77) ~~$\sin\cos^{-1}\left(\cos\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) = b$~~

~~$\sin\cos^{-1}\frac{1}{\sqrt{1+x^2}}$~~

(77) $\tan^{-1}x \in (-\pi/2, \pi/2)$

so $\cos(\tan^{-1}x) \in (0, 1]$

$\cos^{-1}(\cos\tan^{-1}x) \in [0, \pi/2)$

$\Rightarrow \sin\left[\cos^{-1}(\cos\tan^{-1}x)\right] \in [0, 1)$

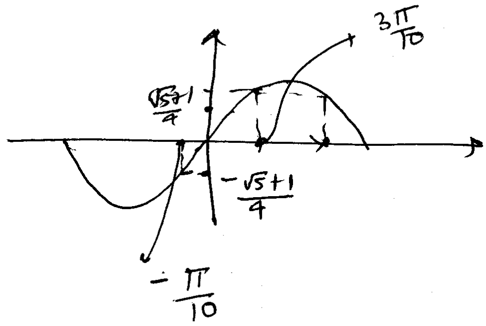
$\Rightarrow b \in [0, 1]$ because both sides are equal.

~~(78)~~

(78) $|4\sin x - 1| < \sqrt{5}$

$\Rightarrow -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$

$-\frac{\sqrt{5}+1}{4} < \sin x < \frac{\sqrt{5}+1}{4}$



$\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$

$\cos\frac{2\pi}{10} = \frac{\sqrt{5}+1}{4}$

$\Rightarrow \sin(\pi/2 - 2\pi/10) = \frac{\sqrt{5}+1}{4}$

$\Rightarrow \sin\frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$

so $x \in \left(-\pi/10, \frac{3\pi}{10}\right)$

(79) $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{\sqrt{(\sqrt2)^2+(\sqrt3)^2-2\sqrt2\sqrt3}}{1+\sqrt2\sqrt3}$

$= \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{\sqrt3-\sqrt2}{1+\sqrt2\sqrt3} = \tan^{-1}\frac{1}{2} - [\tan^{-1}\sqrt3 - \tan^{-1}\sqrt2]$

$= \tan^{-1}\frac{1}{2} + \tan^{-1}\sqrt2 - \tan^{-1}\sqrt3$

$= \tan^{-1}\frac{1}{2} + \tan^{-1}\sqrt2 - \pi/3$

(80) $\lim_{x\to1} \frac{\ln x}{\cos\frac{\pi}{2}x} = \lim_{x\to1} \frac{1/x}{-\frac{\pi}{2}\sin(\frac{\pi}{2}x)} = -\frac{2}{\pi}$

(81) $a \in (0, \frac{1}{2}) \Rightarrow x \in (0, \frac{1}{2})$ Let $x = \tan\theta \Rightarrow \theta \in (0, \pi/6)$

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}\cos 2\theta = 2\theta = 2\tan^{-1}x$

$\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \cot^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = \cot^{-1}(\tan 3\theta)$

$= \pi/2 - \tan^{-1}\tan 3\theta$

$= \pi/2 - 3\theta$

$= \pi/2 - 3\tan^{-1}x$

$\lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$ ($\frac{0}{0}$ form)

Applying L.H. rule.

$\Rightarrow \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{\frac{2}{1+x^2}}{-\frac{3}{1+x^2}} = -3/2$

(82) Apply L.H. rule

$\lim_{x\to2} \frac{f(2)-2f'(x)}{1} = f(2) - 2f'(2)$

(83) Apply L.H. rule.

(84) $\lim_{x\to\infty} \frac{x^{40}(\frac{2}{x}+1)^{40}\ x^5(\frac{4}{x}+1)^5}{x^{45}(\frac{2}{x}-1)^{45}} = \frac{1\cdot1}{(-1)^{45}} = -1$

(85) $\lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2} - \tan^{-1}\frac{x}{x+2}\right]$

$\lim_{x\to\infty} x\left[\tan^{-1}\dfrac{\frac{x+1}{x+2} - \frac{x}{x+2}}{1+\left(\frac{x+1}{x+2}\right)\cdot\frac{x}{x+2}}\right]$

$= \lim_{x\to\infty} x \tan^{-1}\left[\frac{x+2}{2x^2+5x+4}\right]$ $\quad \lim_{x\to\infty} \frac{x+2}{2x^2+5x+4} = 0$

$= \lim_{x\to\infty} x \dfrac{\tan^{-1}\left(\frac{x+2}{2x^2+5x+4}\right)}{\left(\frac{x+2}{2x^2+5x+4}\right)} = 1 \quad \left(\frac{x+2}{2x^2+5x+4}\right)$

$= \lim_{x\to\infty} \frac{x(x+2)}{2x^2+5x+4} = \frac{1}{2}$

(86) L.H.L. $\lim_{x\to 1^-} \frac{x\sin(x-0)}{x-1} = \lim_{x\to 1^-} \frac{x\sin x}{x-1} = -\infty$

R.H.L. $\lim_{x\to 1^+} \frac{x\sin(x-1)}{x-1} = 1$

so ~~that~~ limit does not exist.

(87) $\lim_{x\to 0} \frac{\sin 3x + ax + bx^3}{x^3} = 0$

($\frac{0}{0}$ form)

Applying L.H. rule

$\Rightarrow \lim_{x\to 0} \frac{3\cos 3x + a + 3bx^2}{3x^2} = \left(\frac{3+a}{0}\right)$

$\Rightarrow a = -3$

Sub. a ~~⇒~~

$\lim_{x\to 0} \frac{3\cos 3x - 3 + 3bx^2}{3x^2} = \lim_{x\to 0} \frac{-9\sin 3x + 6bx}{6x}$

$\therefore -27\cos 3x + 6b$

$-\frac{27+6b}{6} = 0 \Rightarrow b = \frac{27}{6} = \frac{9}{2}$

(88) $\cos 4x + \cos 2x + \cos 3x = 0$

$\Rightarrow 2\cos 3x \cos x + \cos 3x = 0$

$\Rightarrow \cos 3x (2\cos x + 1) = 0 \Rightarrow \cos 3x = 0$ or $\cos x = -\frac{1}{2}$

$\cos x = -\frac{1}{2}$ have two sol. in $[0, 2\pi]$ i.e. $2\pi/3$ & $4\pi/3$

$x \in [0, 2\pi]$, $3x \in [0, 6\pi]$

$\cos 3x = 0$, 6 times in $[0, 6\pi]$ (Two times in one period)

Hence total no. of sol. = 8

(89) For P
$\sin\theta - \cos\theta = \sqrt{2}\cos\theta$

$\Rightarrow \sin\theta = (\sqrt{2}+1)\cos\theta$

$\Rightarrow \tan\theta = (\sqrt{2}+1)$

For Q $\quad \sin\theta + \cos\theta = \sqrt{2}\sin\theta$

$\Rightarrow \cos\theta = (\sqrt{2}-1)\sin\theta$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$

For both set P & Q $\tan\theta = \sqrt{2}+1$ so $P = Q$.

(90) $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\alpha} = 1$ & $\lim_{n\to\infty} \sin\frac{1}{n} = 0$

So above limit $= e^{\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{\alpha} + \sin\frac{1}{n} - 1\right]n}$

$= e^{\lim_{n\to\infty}\left\{\left[\left(\frac{n}{n+1}\right)^{\alpha} - 1\right]n + \frac{\sin 1/n}{1/n}\right\}}$

$= e^{\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{\alpha} - 1\right]n + 1}$

$= e^{1}\, e^{\lim_{y\to 0}\left[\left(\frac{1/y}{1/y+1}\right)^{\alpha} - 1\right]/y}$

$= e\, e^{\lim_{y\to 0}\frac{\left(\frac{1}{1+y}\right)^{\alpha} - 1}{y}}$

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